of reference 1 referring to the connection between the relation referred to in that paper by A and our present C is not quite correct without further restriction.)

LEMMA. *If T satisfies (C) and for every $t_1 \epsilon S_2$ there exists a sequence of functions w_N in $C(S_1)$ possibly dependent on the choice of t_1 such that $(Tw_N)(t_1) \rightarrow \infty$ and if $|xy|_t = 0$ then T is multiplicative on xy.*

Suppose $|y|_t = 0$ then

$$|((Txy - Tx\,Ty)Tw_N)(t_1)| \leq \Delta((xy)w_N) + \Delta(x(yw_N)) + \|Tx(Tw_N - TyTw_N)\| \leq \epsilon\,(2 + \|Tx\|).$$

In short $|(Txy - Tx\,Ty)(t_1)| \leq \epsilon\,(2 + \|Tx\|)/|(Tw_N)(t_1)| \rightarrow 0.$

THEOREM 5. *If T satisfies C and is $1 - 1$ on $C(S_1)$ onto $C(S_2)$ then S_1 and S_2 are homeomorphic.*

Indeed since the mapping is "onto" the sequence w_N exists for every choice of t_1 in S_2 and *hence* $|xy|_t = 0$ *implies* $Txy = Tx\,Ty$. Theorem 4 now guarantees T is actually multiplicative and our result follows from Theorem 6 of reference 1 or from reference 3.

[1] Bourgin, D. G., "Approximately Isometric and Multiplicative Transformations on Continuous Function Rings," *Duke Math. J.*, **16**, 385–397 (1949).

[2] Kakutani, S., "Concrete Representation of Abstract M Spaces," *Ann. Math.*, **42**, 994–1024 (1941).

[3] Milgram, A. N., "Multiplicative Semigroups of Continuous Functions," *Duke Math. J.*, **16**, 377–383 (1949).

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*EXTENSIVE GAMES**

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In the mathematical theory of games of strategy as described by von Neumann and Morgenstern,[1] the development proceeds in two main steps: (1) the presentation of an all-inclusive formal characterization of a general n-person game, (2) the introduction of the concept of a pure strategy which makes possible a radical simplification of this scheme, replacing an arbitrary game by a suitable prototype game. These two forms have been given technical names by von Neumann and Morgenstern, who called them the *extensive* and the *normalized* forms of a game. As they have noted, the normalized form is better suited to the derivation of general theorems (e.g., the main theorem of the zero-sum two-person game), while the extensive form exposes the characteristic differences between games and the decisive structural features which determine those differences. Since all games are found in extensive form, while it is practical to normalize but a few, it seems reasonable to attack the completion of a general theory of games in extensive form. This note presents two new results in this theory which appear to have far-reaching consequences in the computational problems of normalized games. These results are cast in terms of a new formulation of the extensive form which seems to have intuitive advantages over that used by von Neumann.[2]

In order to engage in a precise discussion, it will be necessary to clarify certain concepts associated with a game which are confused and ambiguous in common use. We will use these terms in essentially the same manner as von Neumann and Morgenstern. A *game* is simply the set of rules which define it, while every particular instance in which a game is played from beginning to end is called a *play* of that game. A similar distinction is drawn between the occasion of the selection of one among several alternatives, to be made by one of the players or by some chance device, which is called a *move* and the actual selection in a particular play which is called a *choice*. Thus, a game consists of a set of moves in some order (not necessarily linear!), while a play consists of a sequence of choices.

1. The Extensive Form.—

Definition: A general n-person game Γ is a finite tree K imbedded in an oriented plane with the following specifications:

(1) A distinguished vertex O.

(*Terminology:* The *alternatives* at a vertex P are the edges e incident to P and lying in components not containing O if we cut K at P. If there are j alternatives at P, then we index these by the integers $1, \ldots, j$, circling P in the positive sense. At the vertex O, the first alternative may be assigned arbitrarily. If we circle a vertex $P \neq O$ in the positive sense, the first alternative follows the unique edge at P which is not an alternative. Those vertices which possess alternatives will be called *moves;* the remaining vertices will be called *plays.* We define a partition[3] of the moves into sets A_j, $j = 1, 2, \ldots$, where A_j contains all of the moves with j alternatives, which will be called the *alternative partition.*

We also introduce a *temporal order* in the tree K. As in any tree with base point O, there is a uniquely defined unicursal path W_P leading from O to the vertex P. We say that $P \leqq Q$ whenever $P \in W_Q$. This clearly defines a partial ordering of the vertices of K and enables us to assign a (temporal) *rank* to the vertices as follows: O is of rank 1. A vertex P is of rank k if the maximum of the rank of Q such that $Q < P$ is $k - 1$. Using this definition we can introduce the *rank partition* of the moves into sets M_k consisting of all moves of rank k for $k = 1, 2, \ldots$.)

(2) A partition of the moves into $n + 1$ indexed sets $P_0, P_1, \ldots, P_n$ which will be called the *player partition*.

(*Terminology:* The moves lying in P_0 are called *chance moves;* all other moves are called *personal moves.*)

(3) A partition of the moves into sets U which is a refinement of the alternative, player and rank partitions, that is, each U is contained in $P_i \cap A_j \cap M_k$ for some i, j and k. This partition is called the *information partition* and its sets will be called *information sets.*

(4) For each $U \subset P_0 \cap A_j$, a probability distribution on the integers $1, \ldots, j$, which assigns *positive* probability to each. Such U are assumed to be one-element sets.

(5) An n-tuple of real numbers $h(W) = (h_1(W), \ldots, h_n(W))$ for each play W.

(*Terminology:* The function h specified in (5) is called the *pay-off function.*)

The question which must be answered immediately is: How is this formal scheme to be interpreted? That is, how does one play our general n-person game Γ? To personalize the interpretation, one may imagine a number of people isolated from each other with contact with a single person, termed the *umpire.* All persons involved are supposed to know the rules of the game; that is, each is to have a copy of the tree K and the specifications (1)–(5). We assume that there is one person for each information set and that they are grouped into players in the natural manner, a person belonging to the ith player if his information set lies in P_i. This seeming plethora of persons is occasioned by the possibly complicated state of information of our players who may be called upon to forget facts which they knew earlier in a play.[4]

A play begins at the vertex O. We do not exclude the possibility that this is the only vertex in K; then we have a no-move game, no one does anything and the pay-off is $h(O) = (h_1(O), \ldots, h_n(O))$. Suppose that the play has progressed to the move P. Then the play continues by the umpire contacting the person whose information set contains P and, if P is a personal move with j alternatives, asking him to choose a positive integer not greater than j. The person does this, knowing only that he is choosing an alternative at one of the moves in his information set. We assume that the umpire makes all of the chance choices in advance, in accord with the probabilities assigned in (4), so that if P is a chance move then an alternative has already been chosen. In this manner, a path with initial point O is constructed. It is unicursal and hence leads to a play W. (Henceforth we shall utilize the 1–1 correspondence between the plays, which are vertices, and the unicursal paths from O to the plays and use the name play for both objects when no confusion will result.) At this point, the umpire pays player i the amount $h_i(W)$ for $i = 1, \ldots, n$.

A detailed comparison of our formal scheme and von Neumann's axiomatic formulation[5] reveals that if we derive one of our games from a von Neumann game in a natural way the only condition imposed is that all of the plays be of the same rank. This is essentially trivial and can be satisfied by filling out "short" plays with dummy chance moves with only one alternative. Proceeding in the converse direction, if we take one of our games in which all of the plays have the same rank, we impose the single restriction that all of the plays in Ω be admissible under the rules of Γ. Thus we have axiomatized essentially the same set of objects as von Neumann.

2. Pure, Mixed and Behavior Strategies.—Rather than make each decision separately as the occasion demands, a player may devise a plan in advance to cover all possible situations which may confront him. He loses nothing by doing this since he makes his choice a function of the information available to him; consequently, his choice must be constant over each of his information sets. Such a plan is called a *pure strategy.*

Definition: A *pure strategy* π_i for player i in Γ is a choice of a positive integer not greater than j for each set $U \subset P_i \cap A_j$.

If the players choose pure strategies $\pi_1, \ldots, \pi_n$ then a probability p_e is assigned to each alternative e in the graph K; if e is a chance alternative, then p_e is obtained from specification (4), while if e is the νth alternative at a personal move in P_i then $p_e = 1$ if π_i specifies the choice ν on the set U containing this move and $p_e = 0$ otherwise. Clearly, the probability $p_W(\pi_1, \ldots, \pi_n)$ that a play W will occur is given by the formula:

$$p_W(\pi_1, \ldots, \pi_n) = \prod_{e \in W} p_e(\pi_1, \ldots, \pi_n), \tag{6}$$

and hence the expected pay-off to player i is given by

$$H_i(\pi_1, \ldots, \pi_n) = \sum_W p_W(\pi_1, \ldots, \pi_n) h_i(W). \tag{7}$$

Unfortunately, our definition of a pure strategy, while conceptually simple, has an inherent redundancy which we will now eliminate. This redundancy is simple in nature; in the case of a zero-sum two-person game, it is merely the duplication of rows and columns in the pay-off matrix.

Definition: We shall say that two pure strategies for player i are *equivalent*, written $\pi_i \equiv \pi_i'$, if and only if $p_W(\pi_1, \ldots, \pi_i, \ldots, \pi_n) = p_W(\pi_1, \ldots, \pi_i', \ldots, \pi_n)$ for all plays W and all pure strategies for the remaining players, $\pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_n$.

The following definition provides the working criterion for the equivalence of pure strategies.

Definition: A personal move P for player i is called *possible when playing* π_i if there exists a play W and pure strategies for the remaining players $\pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_n$ such that $p_W(\pi_1, \ldots, \pi_i, \ldots, \pi_n) > 0$ and $P \epsilon W$.

Criterion: The pure strategies π_i and π_i' are equivalent if and only if they have the same set of possible moves and specify the same choices on those moves.

Henceforth, when we speak of a pure strategy, we shall mean an equivalence class under the definition just given. Clearly, formulae (6) and (7) still hold, where if e is an alternative specified by a pure strategy on an information set containing a possible move then $p_e = 1$ and $p_e = 0$ otherwise.

However, the simplest games (e.g., Matching Pennies) reveal that a player is at a disadvantage if he uses the same pure strategy in each play. Instead, he can use a probability distribution μ_i on his pure strategies which we will call a *mixed strategy*. Now let π_i appear in the mixed strategy μ_i with probability q_{π_i}. Then the probability that a given play W will result is:

$$p_W(\mu_1, \ldots, \mu_n) = \sum_{\pi_1, \ldots, \pi_n} q_{\pi_1} \cdots q_{\pi_n} p_W(\pi_1, \ldots, \pi_n) \tag{8}$$

and the expected pay-off to player i is given by

$$\begin{aligned} H_i(\mu_1, \ldots, \mu_n) &= \sum_W p_W(\mu_1, \ldots, \mu_n) h_i(W) \\ &= \sum_{\pi_1, \ldots, \pi_n} q_{\pi_1} \cdots q_{\pi_n} H_i(\pi_1, \ldots, \pi_n). \end{aligned} \tag{9}$$

In solving games in normalized form it has been customary to deal with the mixed strategies just introduced. However, instead of mixing pure strategies, a player could specify a probability distribution over the alternatives in each information set and thus plan his action in any given play. We will call the aggregate of such distributions a *behavior strategy.* The advantage of dealing with behavior strategies is a radical reduction of the dimension of the sets involved while the obvious disadvantage derives from a loss of freedom of action. Behavior strategies have been used with telling effect in the solution of individual games by von Neumann,[6] J. Nash and L. S. Shapley,[7] and the author.[8] In our formal treatment we will only deal with behavior strategies which are derived from a mixed strategy.

Definition: Suppose that the mixed strategy μ_i assigns the probability q_{π_i} to each pure strategy π_i and consider the information set U for player i with j alternatives. Let S be the set of pure strategies π_i such that some $P \epsilon U$ is possible when playing π_i. Then S is the disjoint union of the sets $S_1, \ldots, S_j$ where S_ν consists of all π_i which specify the νth alternative on U. If $\sum_{\pi_i \epsilon S} q_{\pi_i} \neq 0$ then we define:

$$b_\nu = \sum_{\pi_i \in S_\nu} q_{\pi_i} \Big/ \sum_{\pi_i \in S} q_{\pi_i} \qquad \text{for } \nu = 1, \ldots, j. \tag{10}$$

If $\sum_{\pi_i \in S} q_{\pi_i} = 0$ then no distribution is defined. The aggregate of all such assignments on information sets U for player i is called the *behavior strategy associated with* μ_i and is denoted by $\beta_i(\mu_i)$ or simply β_i.

Again, if the players choose behavior strategies $\beta_1, \ldots, \beta_n$ then a probability p_e is assigned to each alternative e in the graph K; if e is a chance alternative, then p_e is obtained from specification (4), while if e is the νth alternative at a personal move in P_i at which β_i assigns the probability b_ν then $p_e = b_\nu$ and $p_e = 0$ otherwise. Clearly, the probability $p_W(\beta_1, \ldots, \beta_n)$ that a play W will occur is given by the formula:

$$p_W(\beta_1, \ldots, \beta_n) = \prod_{e \in W} p_e(\beta_1, \ldots, \beta_n) \tag{11}$$

and hence the expected pay-off to player i is given by

$$H_i(\beta_1, \ldots, \beta_n) = \sum_W p_W(\beta_1, \ldots, \beta_n) h_i(W). \tag{12}$$

3. Games with Perfect Information and Games with Perfect Recall.—We shall be concerned with two large classes of games in which the information partition assumes a special form.

Definition: A game Γ is said to have *perfect information*[9] if the information partition consists of one-element sets.

Definition: A game Γ is said to have *perfect recall for player i* if, for all pairs of moves P, Q for player i such that $P < Q$, we have the following condition satisfied. Assume that P and Q lie in the information sets U and V, respectively. Let P have j alternatives and let V_ν be the set of all moves following some $R \in U$ in the temporal order in a play which has the νth alternative at R. Then we demand that $V \subset V_\nu$ for some ν. A game Γ is said to have *perfect recall* if it has such for all players.

The interpretation of these terms is exactly what the names imply. In a game with perfect information, each player is informed at every move of the exact sequence of choices preceding that move. In a game with perfect recall each player remembers everything that he knew and all of his choices at previous moves. The following two theorems hold for such games.

THEOREM 1. *A sufficient condition that an n-person game* Γ *have an equilibrium point*[10] *among the pure strategies for all possible assignments of the pay-off function h is that* Γ *have perfect information.*

THEOREM 2. *A necessary and sufficient condition that*

$$H_i(\beta_1(\mu_1), \ldots, \beta_n(\mu_n)) = H_i(\mu_1, \ldots, \mu_n) \tag{13}$$

for all mixed strategies $\mu_1, \ldots, \mu_n$ *and* $i = 1, \ldots, n$ *in an n-person game* Γ *for all possible assignments of the pay-off function h is that* Γ *have total recall.*

Theorem 1 generalizes the theorem of von Neumann which asserts that a zero-sum two-person game with perfect information is strictly determined. It is proved by the same inductive device with a slight variation due to the absence of the minorant and majorant games in the general n-person case. Theorem 2 enables us to replace mixed strategies by behavior strategies in games with total recall and has many computational ramifications. The proofs of both of the theorems and further considerations of extensive games will be published elsewhere.

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1 Neumann, J. von, and Morgenstern, O., *The Theory of Games and Economic Behavior*, 2nd ed., Princeton University Press, 1947.

2 A graphical representation by a tree has been suggested by von Neumann, *loc. cit.*, p. 77, however he does not treat this matter systematically, preferring a set theoretical formulation.

3 In this paper a partition means an exhaustive decomposition into (possibly void) disjoint sets.

4 It has been noted by von Neumann that Bridge is a two-player game in exactly this manner.

5 Neumann, J. von, and Morgenstern, O., *loc. cit.*, pp. 67–84.

6 Neumann, J. von, and Morgenstern, O., *loc. cit.*, pp. 192–194.

7 Nash, J., and Shapley, L., "A Simple Three-Person Poker Game," *Annals of Mathematics*, Study No. 24 (in preparation).

8 Kuhn, H., "A Simplified Two-Person Poker," *Ibid.*, Study No. 24 (in preparation).

9 Neumann, J. von, and Morgenstern, O., *loc. cit.*, p. 51.

10 Nash, J., "Equilibrium Points in n-Person Games," these PROCEEDINGS, **36**, 48–49 (1950).

THE SPECIFICITY OF ANTI-KIDNEY ANTIBODY DETERMINED BY ITS EFFECT UPON TISSUE CULTURE EXPLANTS

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While investigating the pathogenesis of experimental nephritis produced by rabbit anti-rat-kidney antibody, it occurred to us that the effects of anti-kidney antibody on kidney tissue might readily be visualized in tissue cultures. The specificity of tissue antigens has previously been investigated by the usual immunologic procedures[1] and the effects of antibodies[2, 3] upon tissue explants has long been known. The growth and function of tissue explants have previously been used to study the specificity of tissue