If the cycle in question fails to persist until the value of α corresponding to the given equations is reached, then Theorem I asserts that no part of any cycle of this system lies in the region swept out during the expansion. For any value of α the cycle may become a band of positive width of closed cycles. When α has run through an interval of length π, each cycle must either have disappeared or else have taken the place of another cycle of the original field. Thus the method will always find those limit cycles which enclose one critical point with $\Delta > 0$, and it provides a criterion of non-existence, in case there are no such cycles.

In more complicated situations where the cycle might surround several critical points one must establish the existence of a cycle of the family for some value of α. This may be accomplished by finding a closed curve without contact with the given field (a Bendixson curve), and then choosing the complete family in such a way that this curve becomes a limit cycle for a particular value of α.

THEOREM IV. *All cycles of a given field containing the same critical points as a given Bendixson curve can be constructed by rotation operations of a suitable complete family.*

I wish to thank Professors S. Lefschetz and M. Schiffer for their interest and advice concerning this work.

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*ON THE ADDITION AND MULTIPLICATION THEOREMS FOR SPECIAL FUNCTIONS**

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Communicated by T. Y. Thomas, October 16, 1950

Some years ago I called attention to the unification in the theory of special functions achieved through systematic exploitation of properties of the solutions of the F-equation[1]

$$\frac{\partial F(z, \alpha)}{\partial z} = F(z, \alpha + 1). \tag{1}$$

One virtue of the method is that many formal relationships are discovered and demonstrated in an automatic and trivial fashion, relationships which, if previously known at all, were originally derived at greater length and with more elaborate analytical tools. In particular, if $F(z + y, \alpha)$ be an analytic function of y, we have by Taylor's theorem the addition formula

$$F(z + y, \alpha) = \sum_{n=0}^{\infty} \frac{y^n}{n!} F(z, \alpha + n), \tag{2}$$

which contains as special cases most of the addition formulae and many of the generating expansions for familiar special functions.[2] In this note I wish to point out that this same expansion may be written as a multiplication theorem. First we put $y - z$ for y in (2):

$$F(y, \alpha) = \sum_{n=0}^{\infty} \frac{(y - z)^n}{n!} F(z, \alpha + n). \tag{3}$$

If $y = kz$ then (3) becomes the multiplication theorem

$$F(kz, \alpha) = \sum_{n=0}^{\infty} \frac{(k - 1)^n z^n}{n!} F(z, \alpha + n), \tag{4}$$

which is the desired result.

If, in the general multiplication theorem (4), we put

$$F(z, \alpha) = e^{i\alpha\pi} z^{-\alpha/2} J_{\alpha} (2\sqrt{z}), \tag{5}$$

where $J_\alpha(x)$ is the Bessel function of the first kind, after some reductions we readily obtain the known result[3]

$$J_\alpha(kz) = k^\alpha \sum_{n=0}^{\infty} \frac{(1 - k^2)^n}{n!} \left(\frac{z}{2}\right)^n J_{\alpha+n}(z). \tag{6}$$

If J_α be replaced by Y_α, (6) remains valid. If instead we put

$$F(z, \alpha) = \Gamma(\alpha + 1) (-z)^{-\alpha-1-b} e^{-\frac{1}{z}} L_\alpha^{(b)} ({}^1/z) \tag{7}$$

where $L_\alpha^{(b)}(z)$ is the generalized Laguerre function, by substitution in (5) follows the first multiplication theorem of Erdélyi:[4]

$$k^{\alpha+1+b} e^{-kz} L_\alpha^{(b)} (kz) = \sum_{n=0}^{\infty} \binom{\alpha+n}{n} \left(1 - \frac{1}{k}\right)^n e^{-z} L_{\alpha+n}^{(b)} (z), \tag{8}$$

while if we put

$$F(z, \alpha) = e^{i\alpha\pi} e^{-z} L_b^{(\alpha)} (z), \tag{9}$$

we similarly obtain the second multiplication theorem of Erdélyi:[5]

$$e^{-kz} L_b^{(\alpha)} (kz) = \sum_{n=0}^{\infty} \frac{(1 - k)^n z^n}{n!} e^{-z} L_b^{(\alpha+n)} (z). \tag{10}$$

Of the at least eleven multiplication theorems which hold for the hypergeometric functions, it may be of interest to write down two of the simplest, those resulting from the respective choices

$$F(z, \alpha) = e^{i\alpha\pi} \Gamma(\alpha) z^{-\alpha} F\left(\alpha, c; b; \frac{1}{z}\right), \tag{11}$$

$$F(z, \alpha) = \frac{\Gamma(\alpha - b)\,\Gamma(\alpha - c)}{\Gamma(\alpha)} (1 - z)^{b+c-\alpha} F(b, c;\ \alpha;\ z), \quad (12)$$

namely,

$$F(\alpha, b; c; kz) = k^{-a} \sum_{n=0}^{\infty} \binom{a+n-1}{n} \left(1 - \frac{1}{k}\right)^n F(a + n, b; c; z). \quad (13)$$

$$F(a, b; c; kz) = \left(\frac{1 - z}{1 - kz}\right)^{a+b-c} \sum_{n=0}^{\infty} \left(\frac{k - 1}{1 - \dfrac{1}{z}}\right)^n \binom{1-c+a}{n} \binom{1-c+b}{n} \binom{n}{1-c} F(a, b; c + n; z). \quad (14)$$

Finally, we may put

$$F(z, \alpha) = e^{i\alpha\pi}\,\Gamma(\alpha)\zeta(\alpha, z), \quad (15)$$

where $\zeta(\alpha, z)$ is the generalized Euler-Riemann zeta function, thus obtaining the expansion

$$\zeta(\alpha, kz) = \sum_{n=0}^{\infty} \binom{\alpha+n-1}{n} (1 - k)^n z^n\, \zeta(a + n, z), \quad (16)$$

a result which when $z = 1$ yields a formula for $\zeta(\alpha, k)$ as a series of simple zeta functions.

We conclude with three remarks. The first is that since all the results of this note have been derived by simple substitutions in (2), without use of any limit operation, and since (2) is valid whenever the series on the right converges, it follows that all our results are valid subject only to convergence of the series involved. The second is that since k and z occur symmetrically on the left side of (4), by interchanging them upon the right side and equating the results we obtain the reciprocal theorem

$$\sum_{n=0}^{\infty} \frac{(k - 1)^n z^n}{n!} F(z, a + n) = \sum_{n=0}^{\infty} \frac{(z - 1)^n k^n}{n!} F(k, \alpha + n), \quad (17)$$

subject to the convergence of both series. The third is that by putting $k = 0$ the formula (4) enables us to find

$$\phi(\alpha) = F(0, \alpha) = \sum_{n=0}^{\infty} \frac{(-)^n z^n}{n!} F(z, \alpha + n), \quad (18)$$

a result which is interesting both because the left-hand side is independent of z and because the function $\phi(\alpha)$ occurs explicitly in many of the theorems concerning solutions of the F-equation.

* Prepared under Navy Contract No. N6onr-180, Task Order V, with Indiana University.

[1] Truesdell, C., *An Essay Toward a Unified Theory of Special Functions*, Princeton, 1948.

[2] Op. cit., ¶14.

[3] Watson, J. N., *A Treatise on the Theory of Bessel Functions*, Cambridge, 1922. See ¶5.22.

[4] Erdélyi, A., "Funktionalrelationen mit konfluenten hypergeometrischen Funktionen," I, *Math. Z.*, **42**, 125–143 (1936–1937), eq. (5,3).

[5] Op. cit., eq. (6,2). In this note we employ the definition of the generalized Laguerre function given by E. Pinney, "Laguerre Functions in the Mathematical Foundations of the Electromagnetic Theory of the Paraboloidal Reflector," *J. Math. Phys.*, **25**, 49–79 (1946), so that the lower index is not restricted to integer values. Thus our results (8) and (10) are fully equivalent to Erdélyi's formulae (5,1) and (6,1), which are expressed in terms of the $M_{k,\,m}$ functions of Whittaker. Generalizations of Erdélyi's formulae (5,4) and (6,3) may be obtained either by applying Kummer's first transformation to our expansions (8) and (10) or by inserting successively the two solutions $z^{-\alpha}L_b^{(-\alpha)}(z)/\Gamma(b - \alpha + 1)$ and $\Gamma(\alpha - b)\ (-z)^{-\alpha}L_{-\alpha}^{(b)}\left(\frac{1}{z}\right)$ into the general multiplication theorem (4).