where t_0 is the "refractory" period. The formulation is more complex if the transition is treated as a gradual one or if the "refractory" period has a variable length.

[4] The data reported here were recorded by Mr. Michael Kaplan in the Psychological Laboratories of Columbia University.

[5] This is merely a first approximation. Subsequent analyses may show that the interval between the end of one response and the beginning of the next is not independent of the "holding" period. The results of our procedure indicate that the approximation is useful for the present.

[6] The experiment by Felsinger, Gladstone, Yamaguchi and Hull [*J. Exptl. Psychol.*, **37**, 214–228 (1947)] may not provide an optimal test of our formulation for two reasons. The first is that the data are reported in a frequency distribution with step intervals which begin at zero. If the shortest latency were greater than zero, starting the step intervals at the lowest measure would be more appropriate. The use of zero as a lower limit could easily make an exponential distribution more normal. The method of summarizing the data may account for the deviation of the point at 0.5 second in figure 4. The deviation of this point is an expression of the fact that the distribution reported by Felsinger, Gladstone, Yamaguchi and Hull does not have a maximum frequency at the first step interval.

In the second place, it may be assumed that the many transient discriminative stimuli associated with the exposure of the bar may play a more important rôle than the continuous ones associated with the presence of the bar. Although it is possible to extend the present notion to stimuli of short duration which end before the occurrence of the response, additional assumptions are required. A less equivocal test of the present theory may be expected from a distribution of latencies obtained from an experimental procedure of the sort used by Skinner (op. cit.), Frick [*J. Psychol.*, 26, 96–123 (1948)] and others. After a period of, say, no light, a light is presented and stays on until one response occurs (Skinner) or stays on for some fixed period of time sufficiently long to insure the occurrence of many responses (Frick). Such experimental procedures would minimize unspecified transient stimuli and would parallel more closely the notion that stimulus conditions determine a rate of responding. The procedure used by Frick has the additional advantage of permitting the measurement of the time interval between the onset of the stimulus and the first response and the subsequent intervals between responses under "the same" stimulus conditions.

PARTIAL DIFFERENTIAL EQUATIONS AND GENERALIZED ANALYTIC FUNCTIONS

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1. *Introduction.*—In this note (an abstract of results to be published in full elsewhere) we present the fundamentals of a theory of complex-valued functions $f = u(x, y) + iv(x, y)$ whose real and imaginary parts are connected by the equations

$$u_x = \sigma(x, y)v_y, \; u_y = -\sigma(x, y)v_x, \tag{1}$$

a theory which parallels closely that of analytic functions.

As early as 1891 Picard[1] foresaw the possibility of such a theory, but his suggestion seems to have remained unnoticed. Several years ago Gelbart and the author[2] considered the case $\sigma = a(x)b(y)$. In this case there exists an integration process (used implicitly by Beltrami[3] in a special case) which yields explicit formulas for solutions of (1) corresponding to the powers of a complex variable. From these all other single-valued regular solutions can be constructed, at least locally, by means of "formal power series." Our methods could be extended to systems of higher order (Diaz[4]) and to equations in three-space (Protter[5]), but they seem to be inadequate for the treatment of singular and multiple-valued solutions. The general case (1) has been considered recently by Bergman[6] and by Markushevitch (as reported by Petrovskii[7]), and especially important results have been obtained by Položii.[8] In the present paper we do not construct solutions in closed form as was done in the special case mentioned above, but are able to obtain considerably stronger general results. We shall impose only very mild regularity conditions on the coefficient $\sigma(x, y)$, but in order to achieve as close an analogy as possible with classical function theory we shall require at present that these conditions be satisfied in the whole function-theoretical plane. Most of our results, however, imply theorems of a local character.

It is hardly necessary to point out that the analogy between analytic functions and solutions of partial differential equations of elliptic type has inspired many recent investigations, notably the work of Bergman and Schiffer[9] on the kernel-function, and the theory of mappings by means of solutions of systems of differential equations developed by Lavrentyeff, Schapiro, and Gergen and Dressel.[10]

2. *Pseudo-Analytic Function.*—We set $x + iy = z$ and write functions of x and y as functions of z, without implying by this an analytic dependence on z. A function $\sigma(x, y) = \sigma(z)$ defined in a bounded domain D is called admissible if $\sigma > 0$, and σ_x and σ_y exist and satisfy a Hölder condition. A function σ defined over a Riemann surface F is admissible, if in the neighborhood of each point of F it is admissible as a function of the local parameter. In the following we use a fixed function σ defined and admissible over the whole complex plane (Riemann sphere). Thus $\sigma(z)$ is admissible for $|z| < +\infty$ and $\rho(\zeta) = \sigma(1/\zeta)$ is admissible for $|\zeta| < +\infty$.

We say that a complex-valued function $f(z) = u(x, y) + iv(x, y)$ is pseudo-analytic with respect to σ at a point z_0 if in some neighborhood of this point the partial derivatives of u and v exist, are continuous and satisfy (1). A function is pseudo-analytic in a domain if it is pseudo-analytic at every point in the domain. Complex constants are pseudo-analytic, so are linear combinations of pseudo-analytic functions with real coefficients.

Since equations (1) preserve their form under a conformal transformation, pseudo-analyticity can be defined over an arbitrary Riemann surface.

We shall make almost no use of this fact at present, except for the case of the Riemann sphere.

With each function $g(z) = \varphi(x, y) + i\psi(x, y)$ we associate the "reduced" function ${}^{\sigma(z)}g(z) = \sigma(x, y)^{-1/2}\varphi(x, y) + i\sigma(x, y)^{1/2}\psi(x, y)$. Using this notation we may express pseudo-analyticity by a differentiability requirement.

THEOREM. *A function $f(z)$ defined in a bounded domain D is pseudo-analytic in D if and only if for every z_0 in D the function ${}^{\sigma(z)}[f(z) - f(z_0)]$ (considered as a function of the complex variable z) is differentiable at z_0.*

The proof of this theorem requires much of the theory developed below.

3. *Behavior at a Point.*—The following theorem contains and sharpens some of the recent results by Položii. Our proof uses a method due to Carleman.[11]

THEOREM. *Let $f(z)$ be single-valued, pseudo-analytic and not a constant for $0 < |z - z_0| < r$. Then either $f(z)$ is pseudo-analytic at z_0 and ${}^{\sigma(z)}[f(z) - f(z_0)] \sim a(z - z_0)^n$, where $a \neq 0$ and n is a positive integer, or for some positive integer n, ${}^{\sigma(z)}f(z) \sim a(z - z_0)^{-n}$, $a \neq 0$, or $f(z)$ comes arbitrarily close to any given value in every neighborhood of z_0.*

In the first case we say that $f(z)$ assumes the value $f(z_0)$ with multiplicity n, in the second that $f(z)$ has at z_0 a pole of order n. In the third case (essential singularity) Picard's theorem holds by virtue of a result by Grötzsch,[12] since mappings by pseudo-analytic functions are mappings of bounded eccentricity.

COROLLARY. *Pseudo-analytic functions are interior transformations in the sense of Stoïloff.*

The behavior of a pseudo-analytic function at $z = \infty$ may be described in a similar manner.

4. *Formal Powers.*—Let r be a real rational number, $r \neq 0$, $|r| = p/q$, where p and q are relatively prime positive integers. Let a and ζ be complex numbers, $a \neq 0$. We say that a function $f(z)$ is a formal power with exponent r, coefficient a and center at ζ, and write $f(z) = Z^{(r)}(a, \zeta;\ z)$, if $f(z)$ is q-valued and pseudo-analytic for $0 < |\zeta - z| < +\infty$, if $w = f(z)$ is a one-to-one mapping of the q-times covered z-plane (with branch-points at ζ and ∞) onto the p-times covered w-plane (with branch-points at 0 and ∞) such that $f(\zeta) = 0, f(\infty) = \infty$ if $r > 0, f(\zeta) = \infty, f(\infty) = 0$ if $r < 0$, and if for $z \to \zeta$ we have that ${}^{\sigma(z)}f(z) \sim a(z - \zeta)^r$. We also set $Z^{(0)}(a, \zeta;\ z) \equiv a$, $Z^{(r)}(0, \zeta;\ z) \equiv 0$.

THEOREM. *For every rational r and complex a and ζ the formal power $Z^{(r)}(a, \zeta;\ z)$ exists and is uniquely determined.*

It is easy to see that $Z^{(r)}(a + b, \zeta;\ z) = Z^{(r)}(a, \zeta;\ z) + Z^{(r)}(b, \zeta;\ z)$, and that for every real α, $Z^{(r)}(\alpha a, \zeta;\ z) = \alpha Z^{(r)}(a, \zeta;\ z)$.

With the aid of the formal powers we may associate with every function $f(z)$ which is pseudo-analytic at ζ the "differential quotients"

$$D_\zeta^0 f = f(\zeta),\ D_\zeta^n f = n! \lim_{z \to \zeta} \frac{\overset{\sigma(z)}{\left\{f(z) - \sum_{\nu=0}^{n-1} Z^{(\nu)}(D_\zeta^\nu f/\nu!, \zeta; z)\right\}}}{(z-\zeta)^n},$$

$$n = 1, 2, \ldots.$$

These differential quotients always exist, though $f(z)$ considered as a function of x and y need not have partial derivatives of order higher than the second.

THEOREM. *If $f(z)$ is pseudo-analytic at ζ and $D_\zeta^n f = 0$, $n = 0, 1, \ldots$, then $f(z) \equiv 0$.*

It is easy to see that the uniform limit f of a sequence of pseudo-analytic functions $\{f_j\}$ is pseudo-analytic. It can be shown that if $f_j \to f$ uniformly in a domain D, then $D_\zeta^\nu f_j \to D_\zeta^\nu f$, the convergence being uniform with respect to ζ in every compact subset of D.

5. *Rational Functions.*—We say that a pseudo-analytic function is a rational function if it has no singularities except poles, an entire rational function if it has no singularities except perhaps a pole at infinity.

THEOREM. *Every non-constant rational function has as many zeros as it has poles (provided poles and zeros are counted with proper multiplicities). Every rational function is a sum of formal powers with integral exponents. If $\zeta_i (i = 0, 1, \ldots, R)$ are distinct points, $n_1 (i = 1, 2, \ldots, R)$ non-vanishing integers such that $n_1 + n_2 + \ldots + n_R = 0$ and b a non-vanishing complex number, then there exists one and only one rational function assuming at ζ_0 the value b, having at $\zeta_i (i = 1, 2, \ldots, R)$ a zero of order n_i if $n_i > 0$ and a pole of order $-n_i$ if $n_i < 0$, and having no other zeros or poles.*

If $f(z)$ is an entire rational function and ζ a given complex number, then $f(z)$ may be written as a "formal polynomial with center ζ":

$$f(z) = \sum_{\nu=0}^{n} Z^{(\nu)}(a_\nu, \zeta;\ z). \tag{2}$$

We say that this formal polynomial is of degree n if $a_n \neq 0$. It then has exactly n zeros.

THEOREM. *A formal polynomial of degree n is uniquely determined by its center, its leading coefficient and its n zeros. These parameters may be prescribed arbitrarily.*

Instead of prescribing the zeros of (2) we may require that $f(z)$ should satisfy the conditions $D_{\zeta_j}^\nu f = b_{\nu j}$, $\nu = 0, 1, \ldots, N_j$, $j = 1, 2, \ldots, R$, provided that $\zeta_j \neq \zeta_l$ for $j \neq l$, and $N_1 + N_2 + \ldots + N_R + R = n$.

6. *Formal Power Series and the Expansion Theorem.*—A formal power series is a series of the form

$$\sum_{\nu=0}^{\infty} Z^{(\nu)}(a_\nu, \zeta;\ z). \tag{3}$$

Together with (3) we consider the ordinary power series $\sum a_\nu z^\nu$ and denote its radius of convergence by r.

THEOREM. *The series (3) converges absolutely and represents a pseudo-analytic function for* $|z - \zeta| < r$, *converges uniformly for* $|z - \zeta| \leq r' < r$ *diverges for* $|z - \zeta| > r$.

A similar theorem holds for series of the form $\sum Z^{(-\nu)}$.

THEOREM. *Let* $f(z)$ *be pseudo-analytic for* $|z - \zeta| < R$ *and set* $a_\nu = D_\zeta^\nu f/\nu!$. *There exists a positive number* α *depending only on the function* σ, *such that the series (3) converges and represents the function* $f(z)$ *for* $|z - \zeta| < \alpha R$.

Thus if f is an entire function ($R = +\infty$), the series converges for all finite values of z. A similar expansion theorem holds for functions pseudo-analytic for $|z - \zeta| > R$.

7. *Cauchy's Formula.*—The second order equations resulting from (1) are

$$(u_x/\sigma)_x + (u_y/\sigma)_y = 0, \qquad (\sigma v_x)_x + (\sigma v_y)_y = 0. \qquad (4)$$

Using the so-called fundamental solutions of these equations the values of u and v at an interior point of a domain may be expressed by line-integrals extended over the boundary of the domain and involving the boundary values of u and v (Bergman, Položii).

We are also able, however, without using explicitly the fundamental solutions, to obtain a Cauchy type formula formally identical with the classical one.

We note first that $Z^{(\nu)}(a, \zeta; z)$ depends continuously on a and ζ (at least for $\zeta \neq z$). Hence if C is a rectifiable arc admitting the parametric representation $\zeta = \zeta(s)$, $|\zeta'(s)| = 1$, $s_0 \leq s \leq s_1$, and $A(\zeta)$ a continuous function defined on C, the integral

$$\int_C Z^{(\nu)}[A(\zeta)d\zeta, \zeta; z] = \int_{s_0}^{s_1} Z^{(\nu)}\{A[\zeta(s)]\zeta'(s), \zeta; z\}\, ds$$

is defined for every z not on C.

THEOREM. *Let* **D** *be a domain bounded by a simple closed rectifiable curve* C, $f(z)$ *a function which is pseudo-analytic in* **D** *and continuous on* C. *Then*

$$\frac{1}{2\pi}\int_C Z^{(-1)}\{[\,^{\sigma(\zeta)}f(\zeta)]i d\zeta, \zeta; z\} = f(z)$$

if z *in an interior point of* **D**. *The integral vanishes if* z *is an exterior point of* **D**.

We proceed to enumerate some consequences of this theorem.

8. *Isolated Singularities.*—THEOREM. *Let* $f(z)$ *be single-valued and pseudo-analytic for* $0 < |\zeta - z| < R$. *Then*

$$f(z) = \sum_{\nu=-\infty}^{+\infty} Z^{(\nu)}(a_\nu, \zeta; z),$$

the expansion being convergent for $0 < |\zeta - z| < \alpha R$.

Here α is the constant of the second theorem in §6. The number of non-vanishing $a_{-\nu}$ determines in the usual way the character of the point ζ.

9. *Approximation Theorem.*—Let $\boldsymbol{D}$ be a bounded domain bounded by the closed rectifiable curves $C_0, C_1, \ldots C_N$, C_0 being the outer boundary curve. Let ζ_0 be a point exterior to C_0 (which may be the point at infinity), ζ_j $(j = 1, 2, \ldots, N)$ a point interior to C_j. In $\boldsymbol{D}$ we consider a single-valued pseudo-analytic function $f(z)$.

THEOREM. *There exists a sequence of rational pseudo-analytic functions having no poles except at the points $\zeta_0, \ldots, \zeta_N$, which converges to $f(z)$ in $\boldsymbol{D}$, the convergence being uniform in every compact subset of $\boldsymbol{D}$.*

COROLLARY. *If $\boldsymbol{D}$ is simply connected, f may be expanded in a series of formal polynomials.*

Recently Eichler[13] established the following theorem for certain linear partial differential equations of elliptic type in n-space with analytic coefficients. Let $\boldsymbol{D}_1$, $\boldsymbol{D}_2$, $\boldsymbol{D}_3$ be bounded domains homeomorphic to the interior of the n-sphere, such that $\overline{\boldsymbol{D}}_1 \subset \boldsymbol{D}_2$, $\overline{\boldsymbol{D}}_2 \subset \boldsymbol{D}_3$, $\overline{\boldsymbol{D}}$ denoting the closure of $\boldsymbol{D}$. Let φ be a solution of the equation considered defined in $\boldsymbol{D}_2$ and ϵ a positive number. Then there exists a solution ψ defined in $\boldsymbol{D}_3$ such that $|\varphi - \psi| < \epsilon$ in $\boldsymbol{D}_1$. A more special theorem of this kind has been obtained previously by Bergman.[14] Our result shows that for equations of the form (4) Eichler's theorem holds without analyticity assumptions.

10. *The Logarithmic Function.*—Let ζ' and ζ'' be two distinct points, and set

$$L(a, \zeta', \zeta''; z) = \int_{\zeta'}^{\zeta''} Z^{(-1)}\{[^{1/\sigma(\zeta)}a]d\zeta, \zeta; z\},$$

the integration being performed along some fixed path.

THEOREM. *The function $f(z) = L(a, \zeta', \zeta''; z)$ is pseudo-analytic for $z \neq \zeta', \zeta''$ and is such that the difference $f(z) - [a \log (z - \zeta') - a \log (z - \zeta'')]$ is single-valued and $O(|\log|(z - z')(z - z'')||)$. Any other function with these properties differs from $f(z)$ by a constant.*

From this theorem it is not difficult to infer the existence of a function $f(z) = L(a, \zeta, \infty; z)$ which is pseudo-analytic for $z \neq \zeta, \infty$ and differs from $a \log (z - \zeta)$ by a single-valued function $= O\ (|\log|z - z||)$. If a is real, then $u = \boldsymbol{Re}\, L(a, \zeta', \zeta''; z)$ is a fundamental solution of the first equation (4) for every domain containing ζ' but not ζ''.

11. *Multiple-Valued Functions.*—THEOREM. *Let $f(z)$ be a k-valued pseudo-analytic function defined for $0 < |\zeta - z| < r$. Then*

$$f(z) = \sum_{\nu=-\infty}^{+\infty} Z^{(\nu/k)}(a_\nu, \zeta; z),$$

the expansion being convergent at all points sufficiently near to and distinct from ζ.

If only a finite number of the coefficients $a_{-1}, a_{-2}, \ldots,$ are different from zero, we say that ζ is an algebraic singularity of $f(z)$. Algebraic singularities at $z = \infty$ are defined similarly. A function with only algebraic singularities is called algebraic.

THEOREM. *The sum of all determinations of an algebraic pseudo-analytic function is a pseudo-analytic rational function.*

If $f(z)$ is algebraic, there is associated with it a Riemann surface on which it is single-valued. The genus of this (closed) Riemann surface will be called the genus of $f(z)$.

THEOREM. *Let $f(z)$ be an algebraic pseudo-analytic function of genus p. Then it admits a parametric representation of the form*

$$z = \varphi(t), f = \omega(t)$$

where $\varphi(t)$ is a single-valued analytic function and $\omega(t)$ a single-valued function which is pseudo-analytic with respect to an admissible function $\rho(t)$. If $p = 0$, then $\rho(t)$ is admissible over the whole t-plane, and φ and ω are rational. If $p = 1$, ρ is admissible for $|t| < +\infty$ and all three functions ρ, φ, ω are doubly-periodic with the same periods. If $p > 1$, then ρ is admissible for $|t| < 1$, and all three functions ρ, φ, ω are automorphic with respect to the same group of hyperbolic motions.

While this theorem is almost self-evident (as is the corresponding uniformization theorem for non-algebraic functions) we state it in order to show how the investigation of multiple-valued solutions of (1) leads naturally to the theory of pseudo-analytic functions on Riemann surfaces. This theory should also include equations of the form (1) with multiple-valued $\sigma(x, y)$ and the case where $\sigma(x, y)$ becomes discontinuous or vanishes at certain points and on certain lines.

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