# PARTIAL DIFFERENTIAL EQUATIONS AND GENERALIZED ANALYTIC FUNCTIONS. SECOND NOTE

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Communicated by Marston Morse, November 4, 1950

1. *Introduction.*—This note (an abstract of results to be published in full elsewhere) contains a reformulation and extension of the theory outlined in the preceding note under the same title[1] (henceforth referred to as I). The new theory is consequently based on a complex differentiation process (and on the concomitant integration) and is therefore a true generalization of classical function theory. It applies to all elliptic linear homogeneous partial differential equations of second order (and to systems of first order equations) for unknown functions of two variables. The new theory contains as special cases and unifies the results of I on self-adjoint equations, the earlier more special theory of sigma-monogenic functions by A. Gelbart and the author,[2] and the very ingenious extension of this theory proposed by Markushevitch.[3]

*Notations:* $\mathbf{E}_0$ denotes the finite $z$-plane: $|z| = |x + iy| < +\infty$, $\mathbf{E}$ the whole function theoretical plane (Riemann sphere): $|z| \leq +\infty$. $\bar{\mathbf{D}}$ the closure of a domain $\mathbf{D}$. Functions of the real variables $x$ and $y$ are written as functions of $z = x + iy$, without implying analyticity. If $f$ is a complex number, $\bar{f}$ denotes its complex conjugate. Partial derivatives are denoted by subscripts; in particular, $2w_z = w_x - iw_y$, $2w_{\bar{z}} = w_x + iw_y$. A (complex-valued) function $w(z)$ belongs to class $\mathcal{C}$ (class $\mathcal{H}$) if it is continuous (satisfies a Hölder condition). If $w$ has partial derivatives with respect to $x$ and $y$ of order $n$, which are of class $\mathcal{C}$ (of class $\mathcal{H}$), then $w$ is of class $\mathcal{C}^n$ (of class $\mathcal{H}^n$). $w(z)$ is said to have property $\mathcal{P}$ at $z = \infty$ if $w(1/z)$ has property $\mathcal{P}$ at $z = 0$.

2. *Differentiation and Integration. Pseudo-analytic Functions.*—Two functions, $F(z)$ and $G(z)$, of class $\mathcal{H}^1$ defined in a domain $\mathbf{D}_0 \subset \mathbf{E}_0$ are said to form a *generating pair* in $\mathbf{D}_0$ if $\operatorname{Im}(\bar{F}G) > 0$. Our generalization of function theory is based on assigning to a generating pair $(F, G)$ the part played in the classical theory by $(1, i)$.

A function $w(z)$ defined in a domain $\mathbf{D}$, $\bar{\mathbf{D}} \subset \mathbf{D}_0$, admits the unique representation

$$w(z) = \varphi(z)F(z) + \psi(z)G(z), \tag{1}$$

where $\varphi$ and $\psi$ are real-valued. If for some $z_0 \in \mathbf{D}$ the limit

$$\dot{w}(z_0) = \lim_{z \to z_0} \frac{w(z) - \varphi(z_0)F(z) - \psi(z_0)G(z)}{z - z_0} \tag{2}$$

exists and is finite, $w(z)$ is said to possess at $z_0$ the *(F, G)-derivative* $\dot{w} = d_{(F,G)}w/dz$. It turns out the existence of $\dot{w}$ in a domain has as far-reaching consequences as that of the classical derivative.

A function possessing an $(F, G)$-derivative at all points of $\mathbf{D}$ is called regular *(F, G)-pseudo-analytic of the first kind* (or, more briefly, pseudo-analytic) in $\mathbf{D}$. If (1) is such a function, we call $\omega(z) = \varphi(z) + i\psi(z)$ *pseudo-analytic of the second kind.* The correspondence between $w$ and $\omega$ is one-to-one, and the two functions should be thought of as two representations of the same mathematical entity. The class of pseudo-analytic functions is closed under addition, multiplication by real constants, and uniform convergence.

THEOREM. *If $w(z)$ is pseudo-analytic of the first kind, $w(z)$ and $\dot{w}(z)$ are of class $\mathfrak{C}^1$. If $\omega(z)$ is pseudo-analytic of the second kind, $\omega(z)$ is of class $\mathfrak{C}^2$, and the mapping $\omega = \omega(z)$ is an interior, sense-preserving, quasi-conformal transformation.*

An isolated singularity $z_0$ of a single-valued pseudo-analytic function $w(z)$ is called a pole if $w(z) \sim a(z - z_0)^{-n}$, $z \to z_0$, where $a \neq 0$ and $n$ is a positive integer, an essential singularity if $w(z)$ comes arbitrarily close to every complex number as $z \to z_0$.

THEOREM. *An isolated singularity of a single-valued pseudo-analytic function is either essential, or removable, or a pole.*

Let $W(z)$ be a continuous function, and $\Gamma$ a rectifiable curve leading from $z_0$ to $z_1$. The *(F, G)-integral* of $W$ over $\Gamma$ is defined by the equation:

$$\int_\Gamma W d_{(F,G)} z = F(z_1)\text{Re} \int_\Gamma \frac{\bar{G}W\,dz}{F\bar{G} - \bar{F}G} - G(z_1)\text{Re} \int \frac{\bar{F}W\,dz}{F\bar{G} - \bar{F}G} \quad (3)$$

$W$ is called *(F, G)-integrable* in a domain $\mathbf{D}$ if its $(F, G)$-integral over a closed curve $\Gamma$ vanishes whenever $\Gamma$ is homologous to zero in $\mathbf{D}$.

THEOREM. *A continuous function is (F, G)-integrable if and only if it is an (F, G)-derivative;*

$$\int_{z_0}^{z_1} \dot{w} d_{(F,G)} z = w(z_1) - \varphi(z_0)F(z_1) - \psi(z_0)G(z_1). \quad (4)$$

A generating pair $(F_1, G_1)$ is called a *successor* of $(F, G)$ if $(F, G)$-derivatives are $(F_1, G_1)$-pseudo-analytic, and $(F_1, G_1)$-pseudo-analytic functions are $(F, G)$-integrable. $(F, G)$ is then called a *predecessor* of $(F_1, G_1)$.

THEOREM. *Let (F, G) be a generating pair in $\mathbf{D}_0$. In every bounded domain $\mathbf{D}_1$, $\bar{\mathbf{D}}_1 \subset \mathbf{D}_0$, (F, G) possesses a successor and a predecessor.*

A sequence of generating pairs $\{(F_\nu, G_\nu)\}$, $\nu = 0, \pm 1, \pm 2, \ldots$, is called a *generating sequence* if $(F_{\nu+1}, G_{\nu+1})$ is a successor of $(F_\nu, G_\nu)$. A generating pair $(F, G)$ in $\mathbf{D}_0$ can be embedded (in every bounded domain $\mathbf{D}_1$, $\bar{\mathbf{D}}_1 \subset \mathbf{D}_0$) in a generating sequence $\{(F_\nu, G_\nu)\}$ such that $(F_0, G_0) = (F, G)$. With respect to such a sequence an $(F, G)$-pseudo-analytic function $w(z)$ has *derivatives of all orders:*

$$w^{(0)} = w, \quad w^{(n+1)} = d_{(F_n,G_n)}\, w^{(n)}/dz, \quad n = 1, 2, \ldots \qquad (5)$$

The sequence $\{w^{(n)}(z_0)\}$ determines the function $w(z)$ uniquely.

Two generating pairs, $(F, G)$ and $(\tilde{F}, \tilde{G})$ are called *equivalent* if $\tilde{F} = a_{11}F + a_{12}G$, $\tilde{G} = a_{21}F + a_{22}G$, the $a_{jk}$ being real constants. A generating sequence $\{(F_\nu, G_\nu)\}$ is called periodic if there exists an integer $\mu > 0$ such that $(F_{\nu+\mu}, G_{\nu+\mu})$ is equivalent to $(F_\nu, G_\nu)$. In a bounded domain every generating pair can be embedded in a non-periodic generating sequence. I conjecture, but have not proved, that generating pairs cannot be embedded in periodic sequences, except in special cases.

*Remark:* Since $(F,G)$-differentiability is a conformally invariant property, pseudo-analyticity can be defined on a Riemann surface.

*3. Formal powers.*—A generating pair $(F, G)$ in $\mathbf{E}_0$ is called a generating pair in $\mathbf{E}$ if $F(z)$ and $G(z)$ are of class $\mathcal{H}^1$ at $z = \infty$, and $\mathrm{Im}\{\overline{F(\infty)}G(\infty)\} > 0$.

THEOREM. *A generating pair $(F, G)$ in $\mathbf{E}$ can be embedded in a generating sequence $\{(F_\nu, G_\nu)\}$ in $\mathbf{E}_0$ in such a way that the functions $F_\nu(z)$, $G_\nu(z)$ are continuous at $z = \infty$. The generating pairs $(F_\nu, G_\nu)$ are determined uniquely, up to equivalences.*

In this and the next section we consider a fixed generating pair $(F, G)$ in $\mathbf{E}$ embedded in a sequence described in the theorem. This involves no serious loss of generality, since given a generating pair in $\mathbf{D}_0 \subset \mathbf{E}_0$, we can find a generating pair in $\mathbf{E}$ which coincides with the given one on a preassigned closed subdomain $\overline{\mathbf{D}}_1 \subset \mathbf{D}_0$.

Let $a$ and $z_0$ be complex numbers, $a \neq 0$, and $r$ a real rational number. If $r \neq 0$, set $|r| = p/q$, $p$ and $q$ being relatively prime positive integers; if $r = 0$, set $p = 0$, $q = 1$. An $(F_\nu, G_\nu)$-pseudo-analytic function $w(z)$, $q$-valued and regular for $0 < |z - z_0| < +\infty$ and such that $w(z) \sim a(z - z_0)^r$, $z \to z_0$, and $w(z) = 0(|z|^r)$, $z \to \infty$, is called a (*global*) *formal power* and is denoted by $Z_\nu^{(r)}(a, z_0; z)$. The corresponding function of the second kind is denoted by ${}_*Z_\nu^{(r)}(a, z_0; z)$. For $a = 0$, we set $Z_\nu^{(r)}(0, z_0; z) = 0$.

THEOREM. *The formal powers $Z_\nu^{(r)}(a, z_0; z)$ exist and are unique (for all $\nu, r, a, z_0$). If $a \neq 0$, then the mapping $\omega(z) = {}_*Z_\nu^{(r)}(a, z_0, z)$ is a homeomorphism of the q-times covered z-plane (with branch-points at $z_0$ and $\infty$) onto the p-times covered $\omega$-plane (with branch-points at 0 and $\infty$). $Z_\nu^{(r)}(a, z_0, z)$ is a continuous function of $z_0$. The formal powers satisfy the relations;*

$$Z_\nu^{(r)}(\lambda a + \mu b, z_0; z) = \lambda Z_\nu^{(r)}(a, z_0; z) + \mu Z_\nu^{(r)}(a, z_0; z) \qquad (6)$$

*where $\lambda$ and $\mu$ are real constants, and*

$${}_*Z_\nu^{(0)}(a, z_0; z) = a, \; d_{(F_\nu,G_\nu)}\, Z_\nu^{(r)}(a, z_0; z)/dz = rZ_{(\nu+1)}^{(r-1)}(a, z_0; z). \qquad (7)$$

It follows from (7) that once the sequence $\{(F_\nu, G_\nu)\}$ is known, the formal powers $Z_\nu^{(n)}$, $n = 1, 2, \ldots$, can be obtained by successive integrations.

The same procedure can be carried out starting with an arbitrary generating sequence defined in a bounded domain $\mathbf{D}_0$. In this way we obtain *local formal powers,* which have some properties in common with the global powers. (Local formal powers with negative and/or fractional exponents can also be defined, but not uniquely.)

4. *Applications of Formal Powers.*—With the aid of the global formal powers we can reestablish all results from I §§5-13 in the present more general circumstances. Thus we obtain Taylor and Laurent expansions for single-valued and multiple-valued functions, interpolation theorems for rational (regular but for a finite number of poles) pseudo-analytic functions, the analogs of the fundamental theorem of algebra, of Cauchy's integral formula,[4] of Weierstrass' convergence theorem, of Runge's approximation theorem, etc. We also state a result which describes the totality of single-valued pseudo-analytic functions.

Let $w(z)$ be single-valued and $(F, G)$-pseudo-analytic in a domain $\mathbf{D}$, and let $f(z)$ be single-valued and analytic in this domain. We say that $w(z)$ and $f(z)$ are *similar* if there exists a continuous function $\chi(z)$ in $\overline{\mathbf{D}}$, such that $w(z) = e^{\chi(z)} f(z)$.

THEOREM. *Every single-valued analytic function* $f(z)$, $z \in \mathbf{D}$, *has a similar* $(F, G)$-*pseudo-analytic function* $w(z)$, *and vice versa. If one of the functions is given, the other may be chosen so that* $[w(z)/f(z)] \to 1$ *as* $z \to z_0$, $z_0$ *being a prescribed point of* $\overline{\mathbf{D}}$.

This theorem implies, among other things, the existence of pseudo-analytic functions with infinitely many prescribed isolated zeros and poles.

5. *Pseudo-analytic Functions and Partial Differential Equations.*—In order to express pseudo-analyticity by differential equations we associate with every generating pair $(F, G)$ the functions

$$a = -(\overline{F}G_{\bar{z}} - F_{\bar{z}}\overline{G})/(F\overline{G} - \overline{F}G), \quad b = (FG_{\bar{z}} - F_{\bar{z}}G)/(F\overline{G} - \overline{F}G), \tag{8}$$

$$A = -(\overline{F}G_z - F_z\overline{G})/(F\overline{G} - \overline{F}G), \quad B = (FG_z - F_zG)/(F\overline{G} - \overline{F}G), \tag{9}$$

$$\alpha - i\beta = 4\,(\overline{F}/F)(FG_z - F_zG)/(\overline{F}G - F\overline{G}), \tag{10}$$

$\alpha$ and $\beta$ being real-valued.

LEMMA. *Let* $\mathbf{D}_0$ *and* $\mathbf{D}_1$ *be domains such that* $\overline{\mathbf{D}}_1 \subset \mathbf{D}_0 \subset \mathbf{E}_0$. (a) *Given two functions of class* $\mathfrak{K}$, $a(z)$ *and* $b(z)$, *defined in* $\mathbf{D}_0$, *there exists a generating pair* $(F, G)$ *in* $\mathbf{D}_1$ *such that* (8) *holds.* (b) *Given two real-valued functions of class* $\mathfrak{K}$, $\alpha(z)$ *and* $\beta(z)$, *defined in* $\mathbf{D}_0$, *there exists a generating pair* $(F, G)$ *in* $\mathbf{D}_1$ *such that* (10) *holds.*

The statements (a) and (b) remain valid for $\mathbf{D} = \mathbf{D}_0 = \mathbf{E}_0$ if $a, b, \alpha, \beta = O(|z|^{-2})$, $z \to \infty$.

THEOREM. *A function* $w(z) = \varphi(z)F(z) + \psi(z)G(z)$ *of class* $\mathfrak{C}^1$ *is* $(F, G)$-*pseudo-analytic if and only if*

$$w_{\bar{z}} = aw + b\bar{w} \tag{11}$$

*i.e., if and only if*

$$F\varphi_z + G\psi = 0. \tag{12}$$

*If this condition is satisfied, then*

$$\dot{w} = w_z - Aw - B\bar{w} \equiv F\varphi_z + G\psi_z \tag{13}$$

*and*

$$(\dot{w})_{\bar{z}} = a\dot{w} - B\bar{\dot{w}}. \tag{14}$$

(Note that (14) and part (a) of the lemma imply the existence of successors.)

Eliminating from (12) the function $\varphi$ we obtain the equation

$$\psi_{xx} + \psi_{yy} + \alpha\psi_x + \beta\psi_y = 0 \tag{15}$$

where $\alpha$ and $\beta$ are given by (10).

*Example:* Let $\sigma(z)$ be a positive function of class $\mathfrak{K}^1$, and set $(F, G) = (\sigma^{-1/2}, i\sigma^{1/2})$. Then the $(F, G)$-pseudo-analytic functions of the first (second) kind are the reduced pseudo-analytic (pseudo-analytic) functions of I. If $\sigma = \sigma_1(x)\sigma_2(y)$, then the functions of the second kind are sigma-monogenic.[2] In this case a generating sequence can be obtained by setting $(F_{2\nu}, G_{2\nu}) = (F, G)$, $(F_{2\nu+1}, G_{2\nu+1}) = (\tilde{\sigma}^{-1/2}, i\tilde{\sigma}^{1/2})$, where $\tilde{\sigma} = \sigma_2(y)/\sigma_1(x)$.

Consider now the general linear homogeneous partial differential equation (with real coefficients):

$$A_{11}\psi_{xx} + 2A_{12}\psi_{xy} + A_{22}\psi_{yy} + B_1\psi_x + B_2\psi_y + \Gamma\psi = 0. \tag{16}$$

We assume that the coefficients $A_{jk}$ are of class $\mathfrak{K}^1$, the coefficients $B_j$, of class $\mathfrak{K}$, and that the ellipticity condition $(A_{11}A_{22} - A_{12}^2 > 0)$ is satisfied in the domain considered.[5] By introducing new independent variables (16) can be reduced to the form

$$\psi_{xx} + \psi_{yy} + \alpha\psi_x + \beta\psi_y + \gamma\psi = 0. \tag{17}$$

If $\gamma \equiv 0$, this equation is at the form (15) and it follows from statement (b) of the lemma that, for an appropriate choice of $(F, G)$, the solutions of our equation will be the imaginary parts of $(F, G)$-pseudo-analytic functions of the second kind. If $\gamma \not\equiv 0$, we consider equation (17) in a domain where it possesses a positive solution $\psi_0$, and introducing the new unknown function $\tilde{\psi} = \psi/\psi_0$ reduce the equation to the form (15).

Consider next the general system of linear homogeneous partial differential equations (with real coefficients).

$$A_{j1}\varphi_x + A_{j2}\varphi_y + B_{j1}\psi_x + B_{j2}\psi_y + \Gamma_{j1}\varphi + \Gamma_{j2}\psi = 0, j = 1, 2. \tag{18}$$

This system is elliptic if the roots of the equation $\det(A_{jk} - \lambda B_{jk}) = 0$

are complex. We assume that this condition is satisfied in the domain considered, and that the coefficients $A_{jk}$, $B_{jk}$ are of class $\mathfrak{K}^1$, the coefficients $\Gamma_{jk}$ of class $\mathfrak{K}$. By introducing new independent variables (18) can be reduced to the form

$$\varphi_x = \tau\psi_x + \sigma\psi_y + \gamma_{11}\varphi + \gamma_{12}\psi, \; \varphi_y = -\sigma\psi_x + \tau\psi_y + \gamma_{21}\varphi + \gamma_{22}\psi \quad (19)$$

where $\sigma$ has constant sign. Without loss of generality we assume that $\sigma > 0$. Set $\gamma^2 = \gamma_{11}^2 + \gamma_{12}^2 + \gamma_{21}^2 + \gamma_{22}^2$. If $\gamma \equiv 0$, equations (19) are equivalent to equation (12) for $F = 1, G = -\tau + i\sigma$. Solutions $(\varphi, \psi)$ are then real and imaginary parts of $(F, G)$-pseudo-analytic functions of the second kind. If $\gamma \not\equiv 0$, we reduce (19) to a system of the same form with $\gamma \equiv 0$, by introducing new unknown functions $\tilde{\varphi}$, $\tilde{\psi}$ such that $\varphi = \varphi_1\tilde{\varphi} + \varphi_2\tilde{\psi}$, $\psi = \psi_1\tilde{\varphi} + \psi_2\tilde{\psi}$, where $(\varphi_1, \psi_1)$, $(\varphi_2, \psi_2)$ are fixed solutions of (19) with $\varphi_1\psi_2 - \varphi_2\psi_1 > 0$. The existence of such solutions follows easily from statement (a) of the lemma.

On the other hand, we can introduce new unknown functions $u = \varphi - \tau\psi$, $v = \sigma\psi$. Then $w = u + iv$ satisfies an equation of the form (11). It follows from (a) that, for an appropriate choice of $(F, G)$, solutions $(u, v)$ are real and imaginary parts of $(F, G)$-pseudo-analytic functions of the first kind.

Thus the theory of pseudo-analytic functions bears the same relationship to the general theory of elliptic equations as the theory of analytic functions does to that of the Laplace equation.

[1] Bers, L., *Proc. Natl. Acad. Sci.*, **36**, 130–136 (1950).

[2] Bers, L., and Gelbart, A., *Quart. Appl. Math.*, **1**, 168–188 (1943); *Trans. Am. Math. Soc.*, **56**, 67–93 (1944); *Ann. Math.*, **48**, 342–357 (1947); Bers, L., *Am. J. Math.*, **72**, 705–712 (1950). For a generalization see Lukomskaya, M. A., *Doklady Akad. Nauk*, **73**, 895–888 (1950).

[3] Reported by I. G. Petrovskii, *Uspekhi Mat. Nauk*, **1**, 44–70 (1946).

[4] For other generalizations of Cauchy's formula see Bergman, S., *Trans. Am. Math. Soc.*, **62**, 452–497 (1947); Položii, G. N., *Mat. Sbornik*, **24** (66), 375–384 (1949); Šabat, B. V., *Doklady Akad. Nauk*, **69**, 305–308 (1949).

[5] We do not consider here the interesting case of equations possessing parabolic lines; cf. reference 2, and Weinstein, A., *Trans. Am. Math. Soc.*, **63**, 342–354 (1948); *Quart. Appl. Math.*, **5**, 429–444 (1948).