A "TRY SIMPLEST CASES" RESOLUTION OF THE ABRAHAM-MINKOWSKI CONTROVERSY ON ELECTROMAGNETIC MOMENTUM IN MATTER

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It is the purpose of this communication to establish two conclusions regarding relativistic aspects of electromagnetic momentum in matter and a third conclusion on research methodology, as follows. (1) In *stationary* matter the momentum density is correctly given by Abraham's expression $\mathbf{g}(Ab)$, and not Minkowski's $\mathbf{g}(Mi)$, where[1]

$$\mathbf{g}(Ab) = \mathbf{E} \times \mathbf{H}/c \equiv \mathbf{S}_p/c^2 = \mathbf{g}(Mi)/\epsilon\mu, \qquad (1)$$

where $\mathbf{S}_p$ is Poynting's vector. (2) A medium characterized by nondispersive, lossless, isotropic electrical properties ϵ and μ *in general will not be isotropic for the total energy-momentum tensor components* T_{ij}(Tot) produced by electromagnetic fields: in the specific "cube-stuff" example discussed below, a wave propagating in the $\hat{i}\cos\theta + \hat{j}\sin\theta$ direction in respect to the axes of the simple cubic mechanical structure considered produces in addition to $\mathbf{g}(Ab)$ a kinetic mass motion (or bodily motion) of momentum density $\mathbf{g}_b$ for $\mathbf{H} \parallel \hat{k}$ given by:

$$\mathbf{g}_b = \left[\frac{\epsilon + \mu - 2}{2} - \frac{\mu(\epsilon - 1)^2 \sin^2 2\theta}{4}\right]\mathbf{g}(Ab) - \frac{(\epsilon - 1)^2 \sin 4\theta}{8}\hat{k} \times \mathbf{g}(Ab), \qquad (2)$$

so that the momentum density $\mathbf{g}$(Tot) in the total energy-momentum tensor T_{ij}(Tot) is $\mathbf{g}(Ab) + \mathbf{g}_b$. This example demonstrates the futility of attempts to establish total energy-momentum tensors based solely on considerations of ϵ and μ.[2] (3) The research methodology of this communication deliberately employs the "search thinking tools" recently classified[3] as "try simplest cases" of "conceptual experiments" involving "idealized limiting cases." Use of these tools characterized some of Einstein's foundation-building contributions to relativity;[4] for nearly 60 years they have been overlooked in respect to the electromagnetic energy-momentum tensor, with resultant complexity and confusion. (See, however, *Note added in proof* on Balazs.[16])

Conclusion (1) justifying the $\mathbf{g}(Ab)$ expression can be simply established through center-of-mass considerations reminiscent of Einstein's 1906 paper[4] applied to the conceptual experiment of Figure 1. (The treatment of Fig. 1 was previously presented during a discussion of "hidden momentum" forces on "magnetic currents,"[5] but its logical rigor was not emphasized owing to the authors' unfamiliarity with the relevant relativistic center-of-mass theorems.[1, 4, 6]) To prove equation (1), note that the total momentum associated with power flow P from a source at $\mathbf{r}_0$ to a sink at $\mathbf{r}_0 + \Delta\mathbf{r}$ is

$$\mathbf{G} = P\Delta\mathbf{r}/c^2, \qquad (3)$$

corresponding to the motion of the center of mass in that *uniquely specified inertial system in which matter is stationary* although energy and mass are not. Since $P = IV$, it is then straightforward to verify that in the annular coaxial space the volume integral of $\mathbf{g}(Ab)$ correctly gives $\mathbf{G}$, whereas $\mathbf{g}(Mi)$ gives $\epsilon\mu\mathbf{G}$. (The relationships to microscopic energy-flow considerations are discussed elsewhere.[2, 5, 7, 12] Any worries about mass transport by the conduction electrons can be eliminated by similar considerations applied to a-c power flow through an internally reflecting, "fiber-optics" transmission line formed of a collinear series of right circular $\epsilon\mu$-cylinders with vanishingly small separations. Again the definite relationship of $\mathbf{g}(Ab)$ to power P through $\mathbf{S}_p$ leads to the conclusion that $\mathbf{g}(Ab)$ is correct and $\mathbf{g}(Mi)$ is incorrect.)

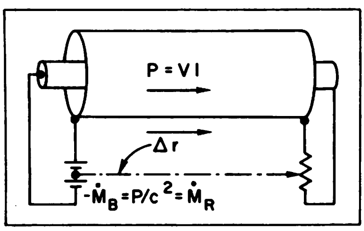

FIG. 1.—Closed system bodily at rest with mass flow from battery to resistor to produce momentum $\mathbf{G} = P\Delta\mathbf{r}/c^2$.

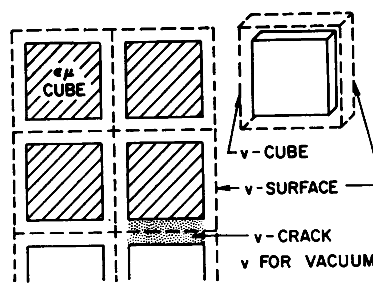

FIG. 2.—"Cube-stuff" composed of "tiny" $\epsilon\mu$-cubes contained in v-cubes (v for vacuum) with v-surfaces in contact in the midplanes of v-cracks between $\epsilon\mu$-cubes.

That a very real difficulty in relativistic electrodynamics has existed for decades in connection with this basic "try simplest cases" result is clearly documented in reviews by Pauli,[8] Møller,[1] and Penfield and Haus.[2] Pauli indicates that experiment would be the correct way to decide between $\mathbf{g}(Ab)$ and $\mathbf{g}(Mi)$ were it not that the predicted difference $(\epsilon\mu - 1)\dot{\mathbf{S}}_p/c^2$ in pondermotive force density is so small that "it is hardly likely that an experiment could be devised for deciding in favor of one or the other of the two approaches." In Note 11, written in 1956, Pauli endorses von Laue's choice of the Minkowski tensor and thus $\mathbf{g}(Mi)$ because it is valid for "the addition theorem of ray velocities."[9] Møller's position is apparently essentially the same, and he quotes Tamm (partially unpublished results) in favor of $\mathbf{g}(Mi)$. Einstein, with co-author Laub,[10] considered pondermotive forces in matter but apparently had not by 1955[11] reached the conclusion presented here regarding the unambiguous correctness of $\mathbf{g}(Ab)$. An experiment to confirm $\mathbf{g}(Ab)$ or at least to reject $\mathbf{g}(Mi)$ is being attempted by R. P. James.[5, 12]

The author's attack on the electromagnetic tensor was a deliberate experiment with the "creative search pattern,"[3] including specifically "action" (ref. 3, 122) of copying equations and deliberate search for simplest, conceptual experiments of idealized limiting cases. The first "pay-off hunch" came within one week and applied ($K3$) and Theorem $\mathbf{F}_c$ (see below) to a set of parallel slabs to obtain the $\theta = 0$ example of equation (2). The electromagnetically isotropic "cube-stuff" model led on the sixth draft of this letter to the unambiguous resolution of the total energy-momentum tensor

$$T_{ij}(\text{Tot}) = S_{ij}(Sh) + B_{ik} \tag{4}$$

into electromagnetic $S_{ij}(Sh)$ and matter B_{ik} parts.

The key attributes of the "cube-stuff" "conceptual experiments" of Figure 2 are: ($K1$) Postulate "tiny": The tiny $\epsilon\mu$-cubes are macroscopic in respect to atoms and microscopic in respect to the scale of observation. ($K2$) The $\epsilon\mu$-cubes, with isotropic, nondispersive, lossless properties, are centered in v-cubes (v for vacuum) packed with their v-surfaces in contact to form a cubic lattice. ($K3$) Postulate "thin cracks": The vacuum spaces in v-cracks between $\epsilon\mu$-cubes are thin in the electrodynamic "crevasse" sense that bulk "cube-stuff" is electromagnetically isotropic with macroscopic fields $\epsilon\mathbf{E} = \mathbf{D}$ and $\mu\mathbf{H} = \mathbf{B}$. ($K4$) Small-signal theory applies, with terms higher than quadratic in $|\mathbf{E}|$ and $|\mathbf{H}|$ being neglected. ($K5$) The Maxwell stress tensor is valid in vacuum.

THEOREM $S_{\kappa\lambda}$. *The fields* $\mathbf{E}_c$ *and* $\mathbf{H}_c$ *within a v-crack with surface having unit normal* $\hat{n}$ *are by* ($K3$)

$$\mathbf{E}_c = \mathbf{E} + (\epsilon - 1)(\mathbf{E}\cdot\hat{n})\hat{n}; \qquad \mathbf{H}_c = \mathbf{H} + (\mu - 1)(\mathbf{H}\cdot\hat{n})\hat{n}, \tag{5}$$

so that by ($K5$) *the stress tensor on the v-surface is*

$$S_{\lambda\kappa} = -E_{c\lambda}E_{c\kappa} - H_{c\lambda}H_{c\kappa} + {}^1/_2\,(\mathbf{E}_c^2 + \mathbf{H}_c^2)\delta_{\lambda\kappa} \tag{6}$$

(*after Møller*,[1] $\lambda,\kappa \leqq 3$), *where* $S_{\lambda\kappa}$ *applies in a v-crack with* $x_\kappa =$ const *so that* $E_{c\lambda} = [1 + (\epsilon - 1)\delta_{\lambda\kappa}]E_\lambda$ *and* $S_{\lambda\kappa} = S_{\kappa\lambda} = -\epsilon E_\lambda E_\kappa - \mu H_\lambda H_\kappa$ *for* $\lambda \neq \kappa$.

THEOREM $\mathbf{F}_c$. *By* ($K5$) *the force on a v-cube is unambiguously obtained by integrating* $S_{\lambda\kappa}$ *over the v-surface, and the result is a rate of change of* $\mathbf{G}_c$, *where* $i\hat{c}\cdot\mathbf{G}_c = \int T_{14}(\text{Tot})\,dV$ *throughout a v-cube.* A consequence of ($K1$) and ($K4$) is that $\mathbf{G}_c$ must be additively composed of a $\mathbf{G}_p$ corresponding to Poynting's vector (ref. 5, eqs. (8), (9), (10), and ref. 7) and $\mathbf{G}_b$ due to kinetic terms (Møller,[1] eq. V-(126)).

THEOREM A-R. *An action-reaction principle applies between contiguous v-cubes since the* $S_{\lambda\kappa}$ *integral on their interface has opposite signs for the two cubes.*

We shall next establish $\mathbf{g}_b$ of equation (2) for a wave propagating in the (100) direction in the lattice, with $\mathbf{E} = \hat{j}E_y$ and $\mathbf{H} = \hat{k}H_z = \hat{k}(\epsilon/\mu)^{1/2}E_y$, and $g_x(Ab) = E_yH_z/c$ so that

$$S_{11} = ({}^1/_2)\,(E_y^2 + H_z^2) = (\epsilon + \mu)E_yH_z/2n = ({}^1/_2)\,(\epsilon + \mu)v\,g_x(Ab), \tag{7}$$

where $v = c/n = c/(\epsilon\mu)^{1/2}$. In initially stationary "cube-stuff," the momentum impulse Δg_x per unit volume produced by arrival of the wave is $-\int(\partial S_{11}/\partial x)\,dt$. That this leads to equation (2) above is established by noting that $v(\partial/\partial x)\,g_x(Ab) = -(\partial/\partial t)\,g_x(Ab)$; we thus obtain

$$\Delta\mathbf{g} = ({}^1/_2)\,(\epsilon + \mu)\,\mathbf{g}(Ab) = \mathbf{g}(Ab) + ({}^1/_2)\,(\epsilon + \mu - 2)\,\mathbf{g}(Ab) = \mathbf{g}(Ab) + \mathbf{g}_b \tag{8}$$

in agreement with equation (2) for $\theta = 0$. As conjectured (ref. 5, footnote 5), the magnetic dual,[13, 14] namely $\mathbf{E} \times \dot{\mathbf{M}}/c = \mathbf{E} \times (\mu - 1)\dot{\mathbf{H}}/c$, to the Lorentz force density $(\epsilon - 1)\dot{\mathbf{E}} \times \mathbf{H}/c$ is essential in explaining electromagnetic momentum. That $\mathbf{g}_b$ is produced by these forces is established by converting the force $\mathbf{F}$ (v-surface) from Theorem $\mathbf{F}_c$ into a volume integral and by using throughout the volume within the surface two vector fields $\mathbf{E}'$ and $\mathbf{H}'$ so defined as to equal $\mathbf{E}_c$ and $\mathbf{H}_c$ in the crack and to change continuously across a negligibly thin surface on the $\epsilon\mu$-cube so as to become $\mathbf{E}$ and $\mathbf{H}$ within it. The result is

$$\mathbf{F}\ (v\text{-surface}) = \int [(\nabla \times \mathbf{E}') \times \mathbf{E}' + (\Delta\cdot\mathbf{E}')\mathbf{E}' + (\nabla \times \mathbf{H}') \times \mathbf{H}' + (\nabla\cdot\mathbf{H}')\mathbf{H}']\, dV$$

$$= \int \left\{\frac{\partial}{\partial t}\mathbf{g}(Ab) + [(\dot{\mathbf{P}} \times \mathbf{H}) + \mathbf{E} \times \dot{\mathbf{M}}]\frac{1}{c} + (\nabla\cdot\mathbf{E}')\mathbf{E}' + (\Delta\cdot\mathbf{H}')\mathbf{H}'\right\} dV. \quad (9)$$

The divergence terms correspond to forces on charges in the surface layer and lead to the θ-dependent parts of $\mathbf{g}_b$ in equation (2); these forces cancel out in pairs for all cubes for (100) propagation with $\theta = 0$. Since $E_y \propto H_z$ for the progressive wave, integrating the $\dot{\mathbf{P}} \times \mathbf{H}$ and $\mathbf{E} \times \dot{\mathbf{M}}$ forces of equation (9) during the time of buildup of $\mathbf{g}(Ab)$ leads to a factor of (1/2) to give $\mathbf{g}_b$ of equation (8).

The need for $\mathbf{g}_b$ in (8) to conserve momentum when a wave enters stationary matter becomes obvious for the "simplest case" of Figure 3: a wave packet

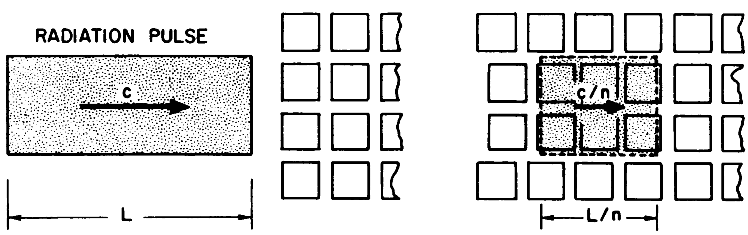

FIG. 3.—Conservation of energy and momentum for an unreflected wave packet incident on "cube-stuff" with $\epsilon = n = \mu$.

incident upon a (100) plane of "cube-stuff" with $\epsilon = n = \mu$ so that no reflection occurs; when the wave enters the matter, the E_y and H_z pattern is unchanged save for compression in length by a factor n. Conservation of energy and momentum can be understood by consideration of the full energy-momentum tensor, including the kinetic terms produced by motion of the $\epsilon\mu$-cubes. Electromagnetic energy, however, is separately conserved since the n-times higher energy density exactly compensates for the smaller volume. However, since $\mathbf{g}(Ab)$ is unchanged, only $1/n$ of the incident momentum remains electromagnetic, a fraction $(n - 1)/n$ being converted to $\mathbf{g}_b$ as given by equation (8). The total energy-momentum tensor is symmetric with $T_{4\lambda} = T_{\lambda 4}$ since the mass flow contributed to $T_{4\lambda}/ci$ by $\mathbf{g}_b$ is also $(n - 1)$ times that carried by Poynting's vector (Møller,[1] IV-(233)). A corresponding increase in T_{44}(Tot) by a factor of n compared to $(\epsilon E^2 + \mu H^2)/2$ occurs owing to increased average density of cubes per unit volume within the wave packet as represented in Figure 3—an obvious consequence of the kinetic energy-momentum tensor θ_{ik} (Møller,[1] eq. IV-(236); eq. V-(123); and remark on $\theta_{\lambda\kappa}$ following equation (12) below). The total energy of the system is conserved by the formation of a surface depression with a resultant loss of mass at the region of incidence. (Is there a laser beam experiment here?)

A summarizing comment on the logic of the use of the "cube-stuff" model is that it is consistent with Maxwell's equations and special relativity and exhibits an internally consistent example of a total energy-momentum tensor. (Cf. Pauli,[8] p. 99.) Any treatment, such as Minkowski's, that on the basis of purportedly general arguments precludes the results of this model must be invalid.

A symmetric tensor that gives that portion of the momentum and energy densities that would remain if the cubes were brought to rest at their lattice sites without changing **E** or **H** is

$$S_{k\lambda}(Sh) = S_{\lambda k}(Sh) = \frac{1}{\epsilon\mu} S_{\lambda k}(Mi) \quad (10a); \qquad S_{4k}(Sh) = S_{4k}(Mi). \quad (10b)$$

This tensor gives the particle velocity[9] behavior of $S_{ij}(Mi)$; furthermore, it satisfies the $\partial S_{ik}(Sh)/\partial x_k = 0$ relationships because the stress tensor terms of the form $S_{12} = -(E_xE_y/\mu) - (H_xH_y/\epsilon)$ are just what is needed by Maxwell's equations, as may be seen by manipulations similar to equation (9).

A prescription for a total energy-momentum tensor T_{ik}(Tot) and the bodily motion part B_{ik} is as follows: Assume initially at $x_4 = ict = 0$ that $B_{44} = -h_0 = -\mu_0 c^2$, the rest-mass density of "cube-stuff," is the only nonvanishing term. Let fields **E** and **H** satisfying Maxwell's equations enter the region; then evaluate $S_{\lambda\kappa}$ and $S_{\lambda\kappa}(Sh)$ from equations (6) and (10) and thus obtain

$$B_{\lambda\kappa} = S_{\lambda\kappa} - S_{\lambda\kappa}(Sh). \quad (11)$$

$B_{\lambda\kappa}$ is the stress tensor that produces bodily momentum $\mathbf{g}_b$ in keeping with Theorem $\mathbf{F}_c$. Next define $B_{\lambda 4}$:

$$B_{\lambda 4} \equiv icg_{b\lambda} = \theta_{\lambda 4} \approx (\epsilon + \mu)|\mathbf{E}||\mathbf{H}|, \quad (12)$$

where θ_{ij} is the kinetic energy-momentum tensor. For small velocities (Møller,[1] eq. IV-(234)), $\theta_{\lambda\kappa} \approx \theta_{\lambda 4}{}^2/\mu_0$ so that by (K4) the kinetic momentum-current tensor is negligible. To obtain B_{4k} let $B_{4\lambda} = B_{\lambda 4}$ and obtain B_{44} by solving

$$\partial B_{4k}/\partial x_k = 0 \quad (13)$$

for B_{44} with $B_{44} = -h_0$ at $t = 0$; evidently B_{44} will contain the extra energy discussed for Figure 3. Thus all the components of B_{ik} and $S_{ik}(Sh)$ and T_{ik}(Tot) of equation (4) are determined: The stress tensor $T_{\lambda\kappa}$(Tot) $= S_{\lambda\kappa}$; $T_{\lambda 4}$(Tot) $= T_{4\lambda}$(Tot) is given by the total momentum density $\mathbf{S}_p/c + \mathbf{g}_b = \mathbf{g}(Ab) + \mathbf{g}_b$. (The addition of mechanical elastic forces to $B_{\lambda\kappa}$ is straightforward; see Fig. 4.)

The fact that T_{ik}(Tot) cannot be determined by ϵ and μ is evident from considerations of waves in other directions than (100). This leads to the results of equation (2). The lack of parallelism between propagation direction and momentum density and hence mass-energy flow results from a transverse displacement of the cubes. Momentum apparently lost from a wave-packet incident, for example, on a (110) surface must be found in the superficial layer of cubes as a consequence of Theorem *A-R*. (Can this surface momentum be detected in laser pulse experiments?) The general behavior of $\mathbf{g}_b$ requires knowledge of $T_{\lambda\kappa}$ that appears to involve previously unexamined, in principle measurable, material constants. (For example, a corrugated *v*-surface can give isotropic $S_{\lambda\kappa}$.)

As a final example of the utility of conceptual experiments with idealized limiting cases,[15] consider a slab of "*springy* cube-stuff" drawn into capacitor plates as shown in Figure 4. A cube in the uniform field region experiences no

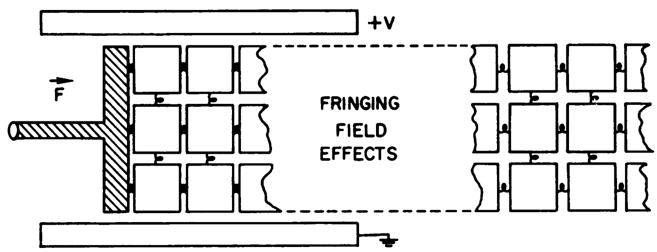

FIG. 4.—The force F exerted on piston to hold a slab of "cube-stuff" from drawing into a charged condenser is entirely transmitted by mechanical springs between the cubes.

net force from $S_{\lambda\kappa}$, so that it is clear that the well-known force exerted on the piston arises in the fringing field region and is "mechanically" transmitted (compare p. 240 of ref. 2) through the mechanical springs connecting cube to cube.

Note added in proof: Balazs,[16] who acknowledged suggestions by Schroedinger, used "try simplest cases" "conceptual experiments" with wave-packets to conclude that $\mathbf{g}$ (*Ab*) is correct and $\mathbf{g}_b$ is necessary; he incorrectly concluded that Abraham's tensor is correct. He did not discuss the force densities $\dot{\mathbf{P}} \times \mathbf{H}/c$ and $\mathbf{E} \times \dot{\mathbf{M}}/c$ of equation (9). The $\mathbf{E} \times \dot{\mathbf{M}}/c$ force density arises from the $\mathbf{m} \times \mathbf{E}/c$ momentum of a stationary current loop that has been recently reviewed by Haus and Penfield[17] and by Coleman and Van Vleck.[18] The relevant history appears to be that the first unambiguous published prediction of the magnetic dual of the Lorentz force was by Costa de Beauregard.[13] The first printing in a journal of an interpretation based on established physical principles was an independent prediction by Shockley and James,[5] who introduced the concept of "hidden momentum" for the momentum associated with relativistic mass flow carried by the matter constituting the amperian-current-loop model of magnetization. A "hidden-momentum" theorem[5, 7] is that any relativistically consistent, steady-state model of an electrically neutral current loop in an electric field must have internal momentum $\mathbf{m} \times \mathbf{E}/c$ through internal mass transport including as examples (1) power flow through mechanical stresses (reminiscent of the now unfashionable Poincaré "hidden momentum" that cancels one fourth of the electromagnetic momentum for the classical electron, see pg. 194, ref. 1), (2) the $\mathcal{E} = mc^2$ mass transport of inertial masses sliding on a circular track, and (3) reversed

Poynting's vector within a magnetic dipole shell. Penfield and Haus had apparently used (2) in lecture notes in 1964, mentioned it obliquely in 1965,[19] and had a description in press in early 1967.[2, 5] A rediscovery of (2) has been formulated in a Darwinian-Lagrangian treatment.[18] Hidden momentum of form (2) was found for the free Dirac electron[12] and occurs for a bound Dirac electron of eigen-energy $\mathcal{E}$ in the form[20] $< \mathbf{p} - (q/c)\mathbf{A} > = < [(\mathcal{E} - q\phi)/c^2]c\alpha >$, which corresponds to mass $\mathcal{E} - q\phi$ flowing with probability current given by $c\alpha$. This analysis thus accords with James' preliminary experimental results that appear definitively to reject **g** (Mi).[21]

The author would like to express appreciation to colleagues at Bell Telephone Laboratories and at Stanford University whose penetrating comments during formative stages of this research led to the focusing on vacuum stress as a key attribute and on a set of parallel ϵ-slabs and μ-slabs as the first "simple case" that exhibited the phenomena of Figure 3. He also appreciates the encouragement of this basic research by the Joint Services Electronics Program at Stanford.

* Under the title "A 'Try Simplest Cases' Resolution of the Energy-Momentum Tensor Controversy for Electromagnetic Fields in Matter."

[1] Møller, C., *The Theory of Relativity* (London: Oxford University Press, reprinted 1966). The units and notation of this communication follow Møller, Greek tensor subscripts $\lambda, \kappa \leqq 3$.

[2] Penfield, Paul, Jr. and Hermann A. Haus, *Electrodynamics of Moving Media* (Cambridge: MIT Press, 1967). These authors also come to the conclusion that ϵ and μ are insufficient. They also conclude that **g** (Ab) is correct. Their analysis, however, uses more sophisticated methods.

[3] Shockley, W., *IEEE Spectrum*, June, 59 (1966); Shockley, W., and W. A. Gong, *Mechanics* (Columbus, Ohio: Charles E. Merrill Books, Inc., 1966). See also *The Conservation of Energy Concept in Ninth Grade General Science*, Project No. S-090, contract OE 6-10-026, U.S. Office of Education, Bureau of Research.

[4] Einstein, A., *Ann. Physik*, **18**, 639 (1905), and **20**, 627 (1906).

[5] Shockley, W., and R. P. James, *Phys. Rev. Letters*, **18**, 876 (1967); *Science*, **156** (3747), 542 (1967).

[6] T. T. Taylor (*Phys. Rev.*, **137**, B467 (1965)) has used center-of-mass consideration to analyze an electromagnetic momentum problem posed by G. T. Trammel (*Phys. Rev.*, **134**, B1183 (1964)).

[7] Shockley, W., *Science*, **158** (3800), 535 (1967).

[8] Pauli, W., *The Theory of Relativity* (New York: Pergamon Press, 1958), Sec. 35.

[9] The necessary and sufficient condition for propagation velocity to have particle velocity transformation laws is that the energy-momentum matrix must have the form $T_{ij} = a_i b_j$, an apparently unrecognized relationship, obvious from Møller eq. VI-(19), and one that is physically necessary for a wave packet to transport momentum components along with its energy. I am indebted to E. I. Blount for pointing out the $a_i b_j$ form of IV-(19).

[10] Einstein, A., and J. Laub, *Ann. Physik*, **26**, 541 (1908).

[11] Einstein, A., *The Meaning of Relativity* (Princeton, N. J.: Princeton University Press, 1955), pp. 49–50.

[12] Shockley, W., *Phys. Rev. Letters*, **20**, 343 (1968).

[13] O. Costa de Beuregard (*Phys. Letters*, **24A**, 177 (1967) (references)) has indicated the need for such a force.

[14] The "hidden momentum" explanation of $\mathbf{E} \times \dot{\mathbf{M}}/c$ has been offered by Shockley and James[5] and independently by Penfield and Haus[2] and shown by Shockley[12] to be consistent with Dirac's equation.

[15] The author's first contribution of this nature was "The empty lattice test of the cellular method in solids," *Phys. Rev.*, **52**, 866 (1937).

[16] Balazs, N. L., *Phys. Rev.*, **91**, 408 (1953).

[17] Haus, H. A., and P. Penfield, Jr., *Phys. Letters*, **26A**, 412 (1968).

[18] Coleman, Sidney, and J. H. Van Vleck, *Phys. Rev.*, in press (July 25, 1968).

[19] Penfield, P., Jr., and H. A. Haus, *Proc. IEEE*, **53**, 422 (1965).

[20] Shockley, W., and K. K. Thornber, personal communication.

[21] James, R. P., personal communication.