FAKE TORI, THE ANNULUS CONJECTURE, AND THE CONJECTURES OF KIRBY*

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Abstract.—The main result of this note (Theorem A) is that the set of piecewise linear (P.L.) manifolds of the same homotopy type as the n-torus, T^n, $n \geq 5$, is in one-to-one correspondence with the orbits of $\Lambda^{n-3}(\pi_1 T^n) \otimes Z_2$ under the natural action of the automorphism group of $\pi_1 T^n$. Every homotopy torus has a finite cover P.L. homeomorphic to T^n; hence the generalized annulus conjecture holds in dimension ≥ 5 (Kirby, R. C., "Stable homeomorphisms," manuscript in preparation). The methods of this classification are also used to study some conjectures of R. C. Kirby (manuscript in preparation) related to triangulating manifolds.

Introduction.—In this note, we shall classify, up to P.L. equivalence, manifolds of the same homotopy type as n-dimensional torus T^n, $n \geq 5$. One of the corollaries of the derivation of this classification can be combined with some recent work of R. C. Kirby[1] to prove the generalized annulus conjecture in dimensions greater than four (see Theorem B and the corollary below). Using the methods and ideas involved in deriving our classification of manifolds of the same homotopy type as T^n, we also study the conjectures of Kirby.[2] They are related to the problem of triangulating manifolds. In particular, one can use our results to obtain information about the homotopy of Top/PL.

C. T. C. Wall has informed us (in a letter to J. L. Shaneson) that he previously proved Theorem A below.

1. *Statement of Results.*—We call a P.L. manifold of the same homotopy as T^n a homotopy torus, and sometimes we call the nonstandard homotopy tori "fake tori." Except for the presence of the fundamental group, T^n is a very simple manifold, perhaps the most simple after the sphere. One might wish that a homotopy torus were always an honest torus. Curiously enough, it is not so. In fact, we have the following result.

Let F be the free **Z**-module of rank n with the automorphism group GL(F). Let GL(F) act on $\Lambda^{n-3}F \otimes Z_2$ in the obvious way, and let B denote the set of orbits. (By $\Lambda^i F$ we mean the ith exterior power of F.)

Theorem A. *There is a one-to-one correspondence between the set of P.L. manifolds of the same homotopy type as $T^n (n \geq 5)$ and the set B.*

For example, if $n = 5$, then there are exactly three homotopy tori (two fake ones and one honest one). The basic reason why such manifolds exist is found in Rohlin's theorem.[4] This is an elaboration of an idea used by J. L. Shaneson and R. Lashof to show that there is a manifold of the same homotopy type as $S^3 \times T^2$ but not P.L.-equivalent.

Theorem B. *Let τ^n, $n \geq 5$, be a P.L. manifold of the same homotopy type as the standard torus T^n. Then τ^n has a finite cover that is P.L. homeomorphic to T^n.*

The proof of this theorem is a straightforward application of the techniques used in proving Theorem A and the definition of Wall's surgery obstructions. We remark that the fake tori still have some finite covers that are fake: see, for example, Remark 4 below.

Now a remarkable new result of Kirby[1] asserts that for $n \geq 5$ the Hauptvermutung for the n-dimensional torus implies that all homeomorphisms of R^n are stable. Hence it implies the annulus conjecture. (The extension of the arguments of ref. 1 to the case $n = 5$ is due to Siebenmann.) In fact it suffices, according to a remark of Siebenmann, to know that every P.L. manifold which is homeomorphic to T^n has a finite cover that is P.L. homeomorphic to T^n. So we obtain the following:

COROLLARY (Generalized Annulus Conjecture). *If $n \geq 5$, every homeomorphism of R^n is stable. Hence, if f and g are disjoint locally flat embeddings of S^{n-1} in S^n, the region between them is homeomorphic to $S^{n-1} \times I$.*

Remarks: (1) There are nonstandard differentiable and P.L.-free $\mathbf{Z}^n$ actions on $\mathbf{R}^n$ ($n \geq 5$) with compact quotient spaces.

(2) Every homotopy torus is smoothable.

(3) No matter what Riemannian structure is put on a fake torus, it is not flat.

(4) A certain infinite cyclic cover of a fake torus M^n is not P.L.-equivalent to $T^{n-1} \times R$ ($n \geq 6$).

(5) If a fake torus M^n is homeomorphic to T^n, then there is a closed topological manifold N^{n+1} which is not of the homotopy type of a P.L. manifold.

In reference 2, Kirby also introduced the following statements:

$S(k,n)$: Let $h:D^k \times T^n \to W^{n+k}$ be a homeomorphism of P.L. manifolds which is P.L. on the boundary. Then h is homotopic to a P.L. homeomorphism that agrees with h on the boundary.

According to reference 2, if $S(k,n)$ is true for all k and n with $k \geq 1$ and $k + n = q \geq 6$, then topological q-manifolds are triangulable. (Ref. 2 refers to stable manifolds, but the *Corollary* above allows us to ignore this difference.) In fact, Siebenmann has noted that it suffices that the conclusion of $S(k,n)$ be valid after passage to a finite cover. We denote the corresponding statement $\tilde{S}(k,n)$.

It is easy to see that in $S(k,n)$ or $\tilde{S}(k,n)$ one can assume, without loss of generality, that h is P.L. on a neighborhood of the boundary. Then, using the simple trick of "digging a hole" near the boundary, one can show that $S(k,n)$, $k \geq 2$, is implied by the following statement:

$T(k,n)$: Let $h: S^{k-1} \times T^n \times I \to S^{k-1} \times T^n \times I$ be a homeomorphism such that $h|S^{k-1} \times T^n \times 0 =$ identity and $f = h|S^{k-1} \times T^n \times 1$ is P.L. Then there is a P.L. homeomorphism of $S^{k-1} \times T^n \times I$ with itself that agrees with $f \cup id$ on $S^{k-1} \times T^n \times \partial I$.

Similarly, the statement $\tilde{T}(k,n)$, in which one allows passage to a finite cover, implies the statement $\tilde{S}(k,n)$.

THEOREM C. *For $k \geq 4$, $k + n \geq 6$, the statement $T(k,n)$ is true. For $n \geq 4$, the statement $\tilde{T}(2,n)$ is true. The statement $\tilde{S}(l,n)$ is true, $n \geq 5$. The statement $T(n,3)$, $n \geq 3$, holds for f^2.*

Remark: The introduction of $T(k,n)$ and $\tilde{T}(k,n)$ is just a convenience to make things fit into the setting of Theorem A. By giving relative versions of the arguments below, one can study the statements $S(k,n)$ and $\tilde{S}(k,n)$ directly.

2. *Indications of the Proofs.*—We start with Theorem A, which is the main result of this note and for which we give the most detailed explanation. Following D. Sullivan,[6] we place ourselves in the following general setting. Let M^n be a closed P.L. manifold. We say (K^n,h) is a homotopy triangulation of M^n if $h:K^n \rightarrow M^n$ is a simple homotopy equivalence, K a P.L. manifold. Two such homotopy triangulations, say $(K,h),(K',h')$, are said to be equivalent if there exists a P.L. equivalence $f:K \rightarrow K'$ such that $h' \circ f$ is homotopic to h. Let $ht(M^n)$ denote the set of equivalence classes of homotopy triangulations. Following reference 6, there is a map $\eta:ht(M^n) \rightarrow [M,F/PL]$.

Now, let $w:\pi_1 M^n \rightarrow \mathbf{Z}_2$ be the homomorphism defined by the first Stiefel Whitney class of M^n, and let $L_i(\pi,w)$ be the ith surgery group of Wall.[7]

A reformulation of the Browder-Novikov theory, essentially the work of D. Sullivan,[3,6] is the following "exact sequence of sets" (cf. also ref. 7, §10).

$$L_{n+1}(\pi,w) \xrightarrow{\partial} ht(M^n) \xrightarrow{\eta} [M^n,F/PL] \xrightarrow{s} L_n(\pi,w). \tag{1}$$

For the case $M^n = T^n$ $(n \geq 5)$, we can show that $ht(T^n) = \eta^{-1}(0)$ or equivalently, $s^{-1}(0) = 0$. Therefore, $ht(T^n) = \partial(L_{n+1}(\mathbf{Z}^n, 0))$.

In order to see the set $ht(T^n)$ clearly, let us recall the map ∂, given by the action of $L_{n+1}(\mathbf{Z}^n, 0)$ on $ht(T^n)$. Let $\alpha \in L_{n+1}(\mathbf{Z}^n, 0)$. Following reference 7, sections 5 and 6, we can construct a manifold W and a degree 1 map $\Phi:W \rightarrow T^n \times I$ including simple homotopy equivalences on the boundaries and a stable trivialization F of $\tau(W) \oplus \Phi^* \epsilon$, with ϵ the trivial bundle such that the surgery obstruction $\Theta[W,\Phi,F] = \alpha$. We define $\alpha(\partial W_-, \Phi|\partial W_-) = (\partial W_+, \Phi|\partial W_+)$ to be the action on $ht(T^n)$.

Let us now identify $\mathbf{Z}^n = \pi_1 T^n$ with a free $\mathbf{Z}$-module F on generators $t_1, \ldots, t_n$ and write $L_{n+1}(\mathbf{Z}^n, 0)$ as $L_{n+1}(F)$. Let J be a subset of $\{1, \ldots, n\}$. We define F_J to be the submodule of F generated by t_i for $i \in J$. Similarly, we define $L_{|J|+1}(F_J)$ in the obvious way.[8] Following reference 5, we define split epimorphisms

$$\alpha_J: L_{n+1}(F) \rightarrow L_{|J|+1}(F_J). \tag{2}$$

It is well known that $L_{|J|+1}(e)$ is a cyclic group.[9] Choose a generator of $L_{|J|+1}(e)$ and denote its image in $L_{|J|+1}(F_J)$ (and then in $L_{n+1}(F)$ by the natural splitting map) under the natural inclusion map by ξ_J. Let Q be the submodule of $L_{n+1}(F)$ spanned by ξ_J with $|J| \neq 3$. We have an isomorphism

$$\Gamma: L_{n+1}(F)/Q \rightarrow \frac{\sum_{i_1<i_2<i_3} L_4(F_{\{i_1, i_2, i_3\}})}{\text{Torsion}} \tag{3}$$

induced by

$$\Sigma\, \alpha_{\{i_1, i_2, i_3\}}$$

We can easily identify $L_{n+1}(F)/Q$ with $\Lambda^{n-3}F$ by using (3). Let us denote the identification by

$$\lambda: L_{n+1}(F)/Q \rightarrow \Lambda^{n-3}F. \quad (5)$$

Note that $GL(F)$ acts on $L_{n+1}(F)$ and $\Lambda^{n-3}F$. We check that Q is invariant under the action. Reducing (5) mod 2, we check that we have a $GL(F)$-equivariant isomorphism

$$\lambda_{(2)}: [L_{n+1}(F)/Q] \otimes \mathbf{Z}_2 \rightarrow \Lambda^{n-3}F \otimes \mathbf{Z}_2. \quad (6)$$

Then, we use the explicit construction of reference 7, sections 5 and 6, and the decomposition formula of $L_{n+1}(F)$ in reference 5 to show that the action of $L_{n+1}(F)$ on $ht(T^n)$ is actually factored through $\Lambda^{n-3}F \otimes \mathbf{Z}_2$ under the identification (6), and there is an onto map

$$g: \Lambda^{n-3}F \otimes \mathbf{Z}_2 \rightarrow ht(T^n). \quad (7)$$

Furthermore, if we let $GL(F)$ act on $ht(T^n)$ in the obvious way, g is equivariant. Finally, we use Rohlin's theorem[4] to show that g is actually one-to-one. Hence, we have Theorem A.

To obtain Theorem B, we start with the fact that by (7) every element of $ht(T^n)$ can be expressed as

$$g\ (\Sigma\ a_{i_1 \ldots i_{n-3}})\ t_{i_1} \Lambda \ldots \Lambda t_{i_{n-3}}, \quad (8)$$

where the coefficients lie in $\mathbf{Z}_2$. Consider, for example, $g(t_1 \Lambda \ldots \Lambda t_{n-3})$. It is not hard to see, from the foregoing analysis, that we have

$$g(t_1 \Lambda \ldots \Lambda t_{n-3}) = \gamma[T^n, id], \quad (9)$$

where $\gamma \in L_{n+1}(F)$ is in the image of $L_{n+1}(F_J)$ under the inclusion-induced map, for any J with $|J| = n - 1$ and $\{n-2, n-1, n\} \subset J$.

By interpreting this fact geometrically as in reference 5 and using the definitions of Wall's surgery obstructions, it is not hard to see that if $h: M \rightarrow T^n$ represents $\gamma[T^n,id]$, then passing to suitable double-covering spaces and covering h by a covering map yields the element $(2\gamma)\cdot[T^n,id]$. The double covers for which this happens are the ones associated with the submodules of F generated by $\{t_1, \ldots, t_{n-3}, 2t_{n-2}, t_{n-1}, t_n\}$, $\{t_1, \ldots, t_{n-2}, 2t_{n-1}, t_n\}$, and $\{t_1, \ldots, t_{n-1}, 2t_n\}$. From the preceding analysis, we have

$$(2\gamma)\cdot[T^n,id] = [T^n,id]. \quad (10)$$

On the other hand, it is also easy to check that if we take a finite cover associated to the submodule of F spanned by $\{b_1 t_1, \ldots, b_{n-3}t_{n-3}, t_{n-2}, t_{n-1}, t_n\}$, we obtain just $\gamma\,[T^n,id]$. The proof of Theorem B now can be concluded by arguing inductively on the length of the expression in (8) and using the fact that addition in $L_{n+1}(F)$ corresponds to gluing up cobordisms in a suitable way. (See ref. 7, or ref. 5, section 1.)

To prove $T(k,n)$, $k \geq 2$, $n + k \geq 6$, it suffices to show that if M_f is the mapping torus of f, f as in $T(k,n)$, then M_f is P.L.-equivalent to $(S^{k-1} \times T^n) \times S^1$. To prove $\tilde{T}(k,n)$, it suffices to show that M_f has a *suitable* finite cover that is P.L.-equivalent to $(S^{k-1} \times T^n) \times S^1$. Using the fact that M_f is homeomorphic to

$(S^{k-1} \times T^n) \times S^1$, we can follow the proof of Theorem A to show that $T(k,n)$ is true, $k \geq 4$, $k + n \geq 6$. We can use the result of reference 6 that if $g; M_f \rightarrow S^{k-1} \times T^n \times S^1$ is a homeomorphism, then $\eta(M_f,g) = 0$; and so $[M_f,g]$ is in the orbit of $id: T^n \rightarrow T^n$ under the action of $L_{n+1}(F)$. In fact, this also gives the following:

PROPOSITION. *Any homeomorphism* $g: M \rightarrow S^{k-1} \times T^{n+1}$, $k + n \geq 6$, $k \geq 5$, *M a* P.L. *manifold, is homotopic to a P.L. equivalence.*

To prove $T(4,n)$, we also have to use the fact that M_f has a naturally embedded copy of $S^3 \times T^n$; this allows us to use Rohlin's theorem again to show that the relevant surgery obstruction in $L_{n+4}(\mathbf{Z}^n)$ must vanish.

To prove $\tilde{S}(1,n)$ and $\tilde{T}(2,n)$, we proceed as in Theorem B. In each case the appropriate mapping torus is homeomorphic to a torus: to T^{n+1} for $\tilde{S}(1,n)$ and to T^{n+2} for $\tilde{T}(2,n)$. In case of $\tilde{S}(1,n)$ (resp. $\tilde{T}(2,n)$), however, there is one (resp. two) generator(s) of the fundamental group that must not be multiplied by 2 (or any other coefficient but 1) when we choose a submodule of F to get a finite cover, as in Theorem B. But this causes us no difficulty because in the proof of Theorem B we had three choices for our finite cover.

For $k = 3$, M_f is homeomorphic to $(S^2 \times T^n) \times S^1$ and has a naturally embedded copy of $S^2 \times T^n$. Using this and the argument used in the proof of Theorem B, one can show that the twofold covering of M_f corresponding to multiplying by 2 the generator of $\pi_1(S^2 \times T^n \times S^1)$ carried by the last S^1 is the standard $(S^2 \times T^n) \times S^1$; here we identify $\pi_1 M_f = \pi_1((S^2 \times T^n) \times S^1)$ via the homeomorphism that the hypotheses of $T(3,n)$ give us. Therefore, $T(3,n)$ is true for f^2.

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[8] If $|J| \leq 3$, we use the periodicity of L_i by setting $L_{|J|+1}(F_J) = L_{4k+|J|+1}(F_J)$ for some k such that $4k + |J| + 1 \geq 6$.

[9] It is equal to Z, Z_2, or 0.