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Pseudo-Poles in the Theory of Emden's Equation

(Thomas–Fermi equation/second-order nonlinear differential equation)

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ABSTRACT **It is known that Emden's equation $y'' = x^{1-m}y^m$ has movable singularities where the solution becomes infinite for one-sided approach. If $m = (p + 2)/p$, p positive integer, the singularities look like poles of order p. In this note expansions in terms of powers and logarithms are obtained from which the nonpolar nature of these "pseudo-poles" becomes evident. Various extensions are considered. Convergence proofs are deferred to a more detailed publication.**

Background

The equation

$$y'' = x^{1-m}y^m,\ 1 < m < 3, \tag{1}$$

is one of two types associated with the name of R. Emden [1]. The special case $m = 3/2$ is the Thomas–Fermi equation [2, 3]. For more recent work on these equations, see E. Hille [3,5] and P. J. Rijnierse [6]. This is a second-order nonlinear differential equation having movable singularities of two distinct types: (*i*) branch points where $y(x)$ is zero and (*ii*) points where y becomes infinite. If m is of the form $m = (p + 2)/p$, p integer >1, the infinitudes look like poles and in the Thomas–Fermi case, $p = 4$, they have been assumed to be poles. If this were the case, the solution would of course be real to both sides of the singularity.

One of the objects of this note is to show that this assumption (at least tacitly made by L. Brillouin [8]) is false: the "pseudo-poles" are not poles at all and a solution can be real to one side of the singularity at most. On the other hand, it must be observed that the dominant part of the expansion at a pseudo-pole is polar and real and the logarithmic perturbation has a damping factor which is $O(|x - c|^{p+2})$.

We concentrate on the case $m = p = 2$, i.e.,

$$y'' = x^{-1}y^2, \tag{2}$$

where the first glimmering of the truth was observed. The resulting pattern was quite clear and more general cases could be handled.

Pseudo-Poles

Eq. (2) has movable singularities of type (*ii*) but not of type (*i*). For second order nonlinear differential equations P. Painlevé determined in the 1890's the equations whose movable singularities are poles. Eq. (2) does not belong to any of the five types of Painlevé (see, e.g. Hille [9], pp. 693–769) so the pseudo-poles are either critical points where the solution is not single-valued or essential singular points. The latter alternative is *a priori* very unlikely. The same conclusions hold for Eq. (1) when $m = (p + 2)/p$. This is the equation

$$y'' = x^{-2/p}y^{(p+2)/p}. \tag{3}$$

Here $p = 2$ gives (2), $p = 4$ the Thomas–Fermi case. Any real nonelementary solution of this equation satisfying certain conditions, say with initial values $y(a) > 0$, $y'(a) \geqslant 0$, has a finite pseudo-pole of order p. That is, there exists a number c, $a < c < \infty$, such that $\lim_{x\uparrow c} y(x) = +\infty$, while

$$y(x) < [p(p + 2)]^{p/2}c(c - x)^{-p} \tag{4}$$

for $x < c$ and this is a best possible estimate. This is a *right* pseudo-pole. Moreover, there are infinitely many solutions having $x = c$ as a right pseudo-pole and satisfying (4). There are also infinitely many solutions having $x = c$ as a *left* pseudo-pole. Here $c - x$ is replaced by $x - c$ and the inequalities are reversed.

To test if this singularity is a pole of order p one substitutes a Laurent series

$$\sum_{n=0}^{\infty} c_n(x - c)^{n-p}. \tag{5}$$

The coefficients c_0 to c_{2p+1} are uniquely determined with $c_0 = [p(p + 2)]^{p/2}c$ as suggested by (4). For the next coefficient one gets

$$0 \cdot c_{2p+2} = M(c_0, c_1, \ldots, c_{2p+1}) \tag{6}$$

where M is a multinomial in the indicated arguments. It has been verified for $p = 2$ and 4 that the right hand side of (6) is not zero so that, at least in these two cases, no Laurent series can exist.

Mixed series expansions

For Eq. (2) a change of independent variable had proved useful for some question and it became a "deus ex machina" also for the pseudo-pole problem. Without restricting the generality we may place the pseudo-pole at $x = 1$, for if $y(x)$ is a solution of (2) and k is any constant, then $ky(kx)$ is also a solution. For a right pseudo-pole at $x = 1$ set

$$x = 1 - e^t, \qquad t < 0. \tag{7}$$

to obtain

$$y''(t) - y'(t) = (1 - e^t)^{-1}e^{2t}[y(t)]^2. \tag{8}$$

Assume a series solution of the form

$$\sum_{n=0}^{\infty} P_n(t)e^{(n-2)t} \tag{9}$$

where P_n is a polynomial in t. This leads to an infinite system of linear second-order nonhomogeneous differential equations for P_n. We have $6P_0 = P_0^2$ or $P_0 = 6$. For $n > 0$

$$P_n''(t) + (2n - 5)P_n'(t) + (n + 1)(n - 6)P_n(t) = \Sigma P_i(t)P_j(t) \equiv Q_n(t) \quad (10)$$

where the summation extends over the integers 0, 1, 2, ..., $n - 1$ with $i + j \leqslant n$ and the combination 0, n to be omitted. Here the coefficients P_1 to P_5 are constants and agree with the corresponding coefficients in (5). For $n = 6$

$$P_6'(t) = A, P_6(t) = At + B. \quad (11)$$

Here A is known in terms of P_0 to P_5 and B is an arbitrary constant. For $n > 6$, the right hand side of (10) is no longer a constant but becomes a polynomial in t of slowly rising degree. The polynomial solution of (10) is of the same degree as Q_n. It is seen that deg $[P_n] = [n/6]$.

Reverting to the original variable and an arbitrary right pseudo-pole at $x = c$ we obtain expansions of the form

$$Y(x; c, B) = \sum_{n=0}^{\infty} P_n[\log (c - x); c, B](c - x)^{n-2} \quad (12)$$

where $P_n(t; c, B)$ is a polynomial in t of degree $[n/6]$ with coefficients that depend upon the two parameters c and B. If B is real the terms are real for $0 < x < c$. Replacing $c - x$ by $x - c$, we obtain similar expansions for a left pseudo-pole. It should be noted that in both cases

$$\sum_{n=0}^{5} c_n(c)(x - c)^{n-2} \quad (13)$$

gives a rational approximation to $Y(x; c, B)$ with an error that is of the order of magnitude of

$$(x - c)^4 \log |x - c|. \quad (14)$$

The approximation is real but the error is real only to one side of $x = c$.

Convergence questions are postponed to a more detailed publication. Here we shall consider only the extension to more general equations. Since (3) is invariant under a suitable transformation $x = as$, $y = bv$, we can normalize c to be 1 and use the transformation (7). The result is

$$y''(t) - y'(t) = e^{2t}(1 - e^t)^{-2/p}[y(t)]^{(p+2)/p} \quad (15)$$

where we set

$$y(t) = \sum_{n=0}^{\infty} P_n(t)e^{(n-p)t}. \quad (16)$$

The analogue of (10) reads

$$P_n''(t) + (2n - 2p - 1)P_n'(t) + (n + 1)(n - 2p - 2)P_n(t) = Q_n(t). \quad (17)$$

Here Q_n is a multinomial in the n arguments $P_0, P_1, \ldots, P_{n-1}$. The coefficients P_0 to P_{2p+1} are constants, P_{2p+2} is linear in t and deg $[P_n] = [n/(2p + 2)]$.

Further generalizations

Actually similar considerations could be applied for a general m, $1 < m < 3$. Suppose $m = (b + 2)/b$ where $b > 1$ and $2b$ is not an integer. Here the situation is much simpler since there are no logarithmic terms at all. It is seen that $y(x)$ has movable singularities where it becomes infinite as $(c - x)^{-b}$. If we replace p by b in (17), it is observed that there is no integer n for which the coefficient of P_n is zero and this means that every P_n is a constant. The solution then has the form

$$\sum_{n=0}^{\infty} c_n(c - x)^{n-b}. \quad (18)$$

Thus, the weird logarithmic case appears iff $2b$ is an integer. Above we have considered only the case where b itself is an integer but the assumption that b is one half of an odd integer leads to the same pattern. In a paper scheduled to appear in the *Unión Matemática Argentina*, the author has considered equations of the form

$$y'' = x^{-5}F(x^3y), \qquad F(u) = \sum_{n=0}^{\infty} F_n u^{\mu_n} \quad (19)$$

with

$$F_n \geqslant 0, \ {}^4/_3 < \mu_0 < \mu_1 < \ldots < \mu_n < \ldots \quad (20)$$

and the series, if infinite, converges for all $u > 0$. If the series breaks off after a finite number of terms, say $F_n = 0$ for $n > p$, then

$$y(x) = O(|x - c|^{-2/(\mu_p - 1)}), \quad (21)$$

that is, the larger μ_p the feebler is the rate of growth. The special case where there is only one term on the right, say

$$y'' = x^{3\mu - 5}y^{\mu} \quad (22)$$

is sufficiently close to the Emden case so we can postulate an expansion

$$\sum_{n=0}^{\infty} P_n[\log (c - x)](c - x)^{n-b} \quad (23)$$

where

$$b = 2/(\mu - 1). \quad (24)$$

The logarithmic terms occur iff $2b$ is an integer. If not, then P_n is a constant. The presence of several terms in the right member of (19) leads to further complications involving an intricate system of exponents besides the logarithmic multipliers. Finally, it should be observed that various physical problems governed by the Thomas–Fermi–Dirac equation have been shown to involve logarithmic perturbations (see chap. 8 of Rijnierse's thesis [7]) but expansions of the type considered here appear to be novel.

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