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# At Least One-Third of Zeros of Riemann's Zeta-Function are on $\sigma = 1/2$

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**ABSTRACT Starting from the derivative of the functional equation of the zeta-function, it is shown that at least one-third of the zeros of the zeta-function are on the line $\sigma = 1/2$.**

It has been shown by Selberg (1) that there exists a computable number $c$ such that the number of zeros of the Riemann zeta-function on the interval $(1/2 + iT, 1/2 + iT + iU)$ is at least $c$ times the total number of zeros in the critical strip in $T < t < T + U$. By a different method it will be shown here that more than $1/3$ of the zeros are on the $1/2$-line. The account here is a sketch. A detailed account will appear later. The account is self-contained. It could also begin from the fact that $\zeta'(s)$ and $\zeta(s)$ have the same number of complex zeros in $\sigma < 1/2$. This latter is a theorem of Levinson and Montgomery, which will appear in a paper entitled "Zeros of the derivatives of the Riemann zeta-function." It has also been pointed out to me by Montgomery that a by-product is that, counting multiplicity, less than 66% of the zeros in the critical strip can be multiple zeros on the $1/2$-line.

Let $h(s) = \pi^{-s/2}\Gamma(s/2)$. Then the functional equation of the Riemann zeta-function is

$$h(s)\zeta(s) = h(1-s)\zeta(1-s). \qquad [1]$$

By Stirling's formula, if $h(s) = \exp(f(s))$ then

$$f(s) = [(s-1)\log(s/2\pi) - s + C_0 + 0(1/s)]/2$$

and

$$h'/h(s) = f'(s) = \log(s/2\pi)/2 + 0(1/s). \qquad [2]$$

Differentiation of Eq. **1** yields

$$h(s)\zeta(s)[f'(s) + f'(1-s)] = -h(s)\zeta'(s) - h(1-s)\zeta'(1-s). \qquad [3]$$

On $s = 1/2 + it$, the right side of Eq. **3** is the sum of two complex conjugates and the zeros of the right side occur where

$$arg[h(1/2 + it)\zeta'(1/2 + it)] \equiv \pi/2 (mod\ \pi). \qquad [4]$$

Since in $|\sigma| < 10$ for large $t$

$$f'(s) + f'(1-s) = \log t/2\pi + 0(1/t),$$

it follows that the zeros due to Eq. **4** are those of $h\zeta(1/2 + it)$ on the left of Eq. **3** and hence are the zeros of $\zeta(1/2 + it)$ itself. By Eq. **1**, if $\chi(s) = h(1-s)/h(s)$, then $\zeta(s) = \chi(s)\zeta(1-s)$ and so

$$\zeta'(s) = -\chi(s)\{[f'(s) + f'(1-s)]\zeta(1-s) + \zeta'(1-s)\}.$$

Thus, by Eqs. **3** and **4** the zeros of $\zeta(1/2 + it)$ occur where

$$arg(h(1-s)\{[f'(s) + f'(1-s)]\zeta(1-s) + \zeta'(1-s)\}) = \pi/2 (mod\ \pi)$$

on $\sigma = 1/2$ or, what is the same thing, where

$$arg(h(s)\{[f'(s) + f'(1-s)]\zeta(s) + \zeta'(s)\}) \equiv \pi/2\ (mod\ \pi) \qquad [5]$$

on the $1/2$-line. Since *arg* $h(s)$ is available from Stirling's formula, it suffices to find the change in the argument of

$$G(s) = \zeta(s) + \zeta'(s)/[f'(s) + f'(1-s)] \qquad [6]$$

on the $1/2$-line. By the formula of Riemann–Siegel (2, ref. 3 section 2.10)

$$\zeta(s) = f_1(s) + \chi(s)f_2(s) \qquad [7]$$

in an obvious notation. From the derivative of Eq. **7**

$$\zeta'(s) = f_1'(s) + \chi(s)f_2'(s) - [f'(s) + f'(1-s)]\chi(s)f_2(s).$$

Using this and Eq. **7** in Eq. **6**

$$G(s) = H(s) + H_1(s) \qquad [8]$$

where

$$H(s) = f_1(s) + [f_1'(s) + \chi(s)f_2'(s)]/\log(t/2\pi), \qquad [9]$$

$$H_1(s) = [f_1'(s) + \chi(s)f_2'(s)]0(t^{-1}\log^{-1} t/2\pi). \qquad [10]$$

To get the change in *arg* $G(s)$ on the $1/2$-line, the principle of the argument can be used. The determination of the number of zeros of $G(s)$ in a rectangle $D$ with vertices $(1/2 + iT, 3 + iT, 1/2 + i(T+U), 3 + i(T+U))$ leads in a familiar way to the change in *arg* $G$ on $\sigma = 1/2$, $T < t < T + U$. Here $U = T/\log^{10} T$. To get the number of zeros of $G(s)$ in $D$, Littlewood's lemma (ref. 3, section 9.9) is used in familiar way (ref. 3, section 9.15) and the crucial term for the number of zeros in $D$ is, for $a < 1/2$ and $1/2 - a$ very small,

$$\int_T^{T+U} \log|G(a+it)|dt/2\pi(1/2 - a). \qquad [11]$$

Actually in a well-known way (1), $G$ is first multiplied with a modifier which can only increase the number of zeros in $D$ but makes the analysis tractable. Let $\Sigma$ here be for $1 \le j \le y$ and let $\psi(s) = \Sigma b_j/j^s$, $y = T^{1/2}/\log^{40} T$, $b_j = \mu(j)\log y/j/(j^{1/2-a}\log y)$. Instead of the numerator in Eq. **11**,

$$\int_T^{T+U} \log|G\psi(a+it)|dt$$

will be used and this is in turn dominated by

$$(1/2)U \log\left[U^{-1}\int_T^{T+U} |G\psi(a+it)|^2 dt\right]. \qquad [12]$$

Because of Eq. **10** it is a triviality that $G$ can be replaced in Eq. **12** by $H$ and, hence, the crucial calculation involves

$$\int_T^{T+U} |H\psi(a+it)|^2 dt. \qquad [13]$$

The following are well known:

$$\chi = (t/2\pi)^{1/2-\sigma} \exp(\pi i/4 - it\log t/2\pi e)(1 + 0(t^{-1}))$$

$$f_1(s) = \sum_{j\le(t/2\pi)^{1/2}} j^{-s} + 0(t^{-\sigma/2})$$

$$f_2(s) = \sum_{j\le(t/2\pi)^{1/2}} j^{s-1} + 0(t^{(\sigma-1)/2}).$$

From Eq. **9**,

$$\int_T^{T+U} |H\psi(a+it)|^2 dt = I_{11} + I_{22} + I_{33} + 2(I_{12} + I_{13} + I_{23})$$

where

$$I_{11} = \int_T^{T+U} |f_1\psi(a+it)|^2 dt$$

$$I_{12} = Re\int_T^{T+U} f_1\bar{f}_1{}'\,\psi\bar{\psi}(a+it)dt/\log t/2\pi$$

$$I_{33} = \int_T^{T+U} |\chi f_2{}'\,\psi(a+it)|^2 dt/\log^2 t/2\pi \text{ etc.}$$

Let $T_1 = \max(T, 2\pi j_1^2, 2\pi j_2^2)$. Then, after some calculation,

$$I_{11} \sim \sum b_{k_1}b_{k_2}k_1^{-a}k_2^{-a}\sum j_1^{-a}j_2^{-a}\int_{T_1}^{T+U}\exp(it\log j_2k_2/j_1k_1)dt$$

where the outer sum is for $k_1,\ k_2 \le y$ and the inner one is for $j_1, j_2 \le \tau$ where $\tau = (T/2\pi)^{1/2}$. If $k = (k_1,k_2)$, $k_1 = A_1k$, $k_2 = A_2k, j_1 = A_2j,\ j_2 = A_1j$, and $A_M = \max(A_1,A_2)$ then

$$I_{11} \sim U\Sigma b_{k_1}b_{k_2}k^{2a}\,k_1^{-2a}\,k_2^{-2a}\,\Sigma j^{-2a}$$

where the inner sum is for $j \le \tau/A_M$. Let $k_M = \max(k_1,k_2)$ and $k_m = \min(k_1,k_2)$. From the above follows

$$I_{11} \sim U\tau^{1-2a}/(1-2a)\Sigma b_{k_1}b_{k_2}kk_M{}^{-1}k_m{}^{-2a} - U/(1-2a)\Sigma b_{k_1}b_{k_2}k^{2a}\,k_1^{-2a}\,k_2^{-2a}.$$

Let $L$ denote $\log T/2\pi$. Similar computations lead to

$$\begin{aligned}I_{22} \sim\ & U\tau^{1-2a}(2-4a)^{-1}L^{-1}\Sigma b_{k_1}b_{k_2}kk_M{}^{-1}k_m{}^{-2a}\log \tau k_m/k_M\\ &- U\tau^{1-2a}(1-2a)^{-2}L^{-2}\Sigma b_{k_1}b_{k_2}kk_M{}^{-1}k_m{}^{-2a}(L/2 + \log \tau k_m/k_M)\\ &+ 2U\tau^{1-2a}(1-2a)^{-3}L^{-2}\Sigma b_{k_1}b_{k_2}kk_M{}^{-1}k_m{}^{-2a}\\ &- U(1-2a)^{-1}L^{-2}\Sigma b_{k_1}b_{k_2}k^{2a}\,k_1^{-2a}\,k_2^{-2a}\log k_1/k\log k_2/k\\ &+ 2U(1-2a)^{-2}L^{-2}\Sigma b_{k_1}b_{k_2}k^{2a}\,k_1^{-2a}\,k_2^{-2a}\log k_1/k\\ &- 2U(1-2a)^{-3}L^{-2}\Sigma b_{k_1}b_{k_2}k^{2a}\,k_1^{-2a}\,k_2^{-2a}\end{aligned}$$

and similarly for the other $I$'s. The results for $I_{13}$ and $I_{23}$ are much more complicated to compute and involve the use of the method of stationary phase (ref. 1, Lemma 2). For $I_{13}$ and $I_{23}$, the use of polar coordinates at an appropriate point in the calculation also is convenient. Fortunately, when all the terms are added the asymmetric terms cancel and there results

$$\begin{aligned}\int_T^{T+U} |H\psi(a+it)|^2 dt \sim\ & U\Sigma b_{k_1}b_{k_2}k^{2a}k_1^{-2a}k_2^{-2a}[-(1-2a)^{-1}\\ &- (1-2a)^{-1}L^{-2}\log k_1/k\log k_2/k + 2(1-2a)^{-2}L^{-2}\\ &\times \log k_1/k - 2(1-2a)^{3}L^{-2} + 2(1-2a)^{-1}L^{-1}\log k_1/k\\ &- 2(1-2a)^{-2}\,L^{-1}] + U\tau^{2-4a}\Sigma b_{k_1}b_{k_2}k^{2-2a}\,k_1^{-1}\,k_2^{-1}\\ &\times [(1-2a)^{-1}L^{-2}\log k_1/k\log k_2/k + 2(1-2a)^{-2}L^{-2}\\ &\times \log k_1/k + 2(1-2a)^{-3}L^{-2}].\end{aligned} \qquad [14]$$

Because of the occurrence of $\tau^{2-4a}$, the last terms are the more important. Let

$$K_{11} = \Sigma b_{k_1}b_{k_2}k^{2-2a}\,k_1^{-1}\,k_2^{-1}.$$

Then from the definition of $b_j$

$$K_{11} = \log^{-2} y\ \Sigma\mu(k_1)\mu(k_2)\ k^{2-2a}\ k_1^{-3/2+a}\ k_2^{-3/2+a} \times \log y/k_1\log y/k_2$$

since

$$\sum_{r|m}\mu(r)(m/r)^{2-2a} = m^{2-2a}\,\theta_{2-2a}(m)$$

where

$$\theta_{2-2a}(m) = \prod_{p|m}(1 - 1/p^{2-2a}), \qquad [15]$$

$$\begin{aligned}K_{11} &= \log^{-2} y\ \Sigma j^{2-2a}\,\theta_{2-2a}(j)\Big(\sum_{j|k_1}\mu(k_1)\ k_1^{-3/2+a}\log y/k_1\Big)^2\\ &= \log^{-2} y\ \Sigma j^{-1}\,\theta_{2-2a}(j)\mu^2(j)(\Sigma'\mu(n)\ n^{-3/2+a}\log y/nj)^2\end{aligned}$$

where $\Sigma'$ is for $n \le y/j$ and $(n,j) = 1$. Let $x = y/j$ for the moment and let $b$ be real and near 1. Let

$$S = \Sigma'\mu(n)\ n^{-b}\log x/n. \qquad [16]$$

Then

$$2\pi iS = \int x^{w-b}(w-b)^{-2}/(\zeta(w)\Pi'(1 - 1/p^w))dw$$

where $\Pi'$ is for $p|j$. The line of integration is on $Re\ w = 2$. Then by residue theory and appraisals on a shifted contour

$$2\pi iS \sim [(b-1)\log x + 1]/\theta_b(j)$$

very crudely stated, which does yield

$$K_{11} \sim \log^{-2} y\ \Sigma j^{-1}\,\theta_{2-2a}(j)\mu^2(j)\,[(1/2 - a)\log y/j + 1]^2/$$

$$\theta^2_{3/2-a}(j) \sim (1/2 - a)^2(\log y)/3 + (1/2 - a) + (\log y)^{-1}.$$

The other sums $K_{12}$ and $K_{22}$ are treated similarly where $K_{12}$ involves the additional factor $\log k_1/k$ and $K_{22}$ contains $\log k_1/k$ $\log\ k_2/k$.

Let $R = (1-2a)\log\tau$ and recall $\log y \sim \log\tau$. Then the above computation leads to

$$Ue^{2R}(R^{-3}/2 + R^{-1}/24) \qquad [17]$$

for the terms involving $\tau^{2-4a}$. Similar treatment of the other sums in Eq. **14** leads finally to

$$U^{-1}\int_T^{T+U} |H\psi(a+it)|^2 dt \sim e^{2R}(R^{-3}/2 + R^{-1}/24) - R/12 + 7/12 - 25/(24R) - R^{-2} - 1/(2R^3). \qquad [18]$$

If $R = 1.3$, an elementary computation shows that the right side of Eq. **18** is dominated by 2.3502. Since $H$ can replace $G$ in Eq. **12**, it follows that

$$\int_T^{T+U} \log |G\psi(a + it)| dt \leq U(\log 2.3502)/2 \leq .4280U$$

and so

$$({}^1/_2 - a)^{-1} \int_T^{T+U} \log |G\psi(a + it)| dt \leq$$
$$.4280U(\mathrm{Log}\ T/2\pi)/R < .3293U\ \mathrm{Log}\ T/2\pi.$$

Accordingly, the change in $arg(G({}^1/_2 + it)$ on $(T, T + U)$ is at most $.33U$ Log $T/2\pi$. (The case of zeros of $G$ on the ${}^1/_2$-line requires a little added discussion.) By Stirling's formula, the change in $arg\ h({}^1/_2 + it)$ is essentially ${}^1/_2U$ Log $T/2\pi$ and so the change in $arg\ hG({}^1/_2 + it)$ is at least $.17U \log T/2\pi$. Since the zeros occur mod $\pi$, the number of them is at least $.34U$ $(\log T/2\pi)/2\pi$, which does in fact exceed ${}^1/_3$ the number of zeros of the zeta-function in the critical strip for $T < t < T + U$ and proves the result.

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1. Selberg, A. (1942) "On the zeros of Riemann's zeta function," *Skr. Norske Vid. Akad. Oslo* **10**, 1–59.
2. Siegel, C. L. (1932) "Über Riemann's Nachlass zur analytischen Zahlentheorie," *Quellen und Studien zur Geschichte der Math. Astr. und Physik, Abstr. B* **2**, 45–80.
3. Titchmarsh, E. C. (1951) *The Theory of the Riemann Zeta-Function* (Oxford Univ. Press, London).