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An extension of the concept of inertial frame and of Lorentz transformation*

(projective relativity/5-dimensional rotations)

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ABSTRACT It is shown how particular kinds of fractional-linear (or projective) transformations generalize the notion of inertial frame in that they ensure that free-particle motion goes over into free-particle motion. A ten-parameter group of such transformations is produced which generalize Lorentz transformations, and which involve besides c (velocity of light) a new fundamental length b; they encompass the ordinary Lorentz group in the limit that b becomes infinite. These extended Lorentz transformations are most simply understood as a type of rotation in the space of homogeneous coordinates, a rotation that unifies 3-space rotations, frame-shifts to moving frames, and space- as well as time-translations. The structure of the invariant differential line element and of the wave operator that generalize those of special relativity are discussed, and implications for the possible revision of usual physical statements are pointed out.

Perhaps no concept in physics is as fundamental as that of inertial frame, understood since Newton to mean a frame in which a "free" particle runs in uniform rectilinear motion. Having one such frame (a prototype like that set by the fixed stars) all others—the ordinary inertial frames or OIF—are generally thought to be obtainable by linear transformations, such as Galilean or Lorentz transformations, since these self-evidently preserve the linearity of free-particle motion.

In the present note, we remark on a family of transformations, including but going beyond the OIF, that preserve free-particle motion and can be properly called extended inertial frames or EIF. They form a group and can be specialized to ensure a central role to the velocity of light c while bringing in a new fundamental universal length constant b in such a way that as $b \to \infty$ the ordinary Lorentz transformations (OLT) are recovered. These extended Lorentz transformations (ELT) are indeed a particular form of projective (or fractional-linear or homographic) transformations. They contain 10 parameters (rather than the 24 of the general projective group) associated to the covariance of physical statements under space-rotations, frame-changes to relatively moving systems (or boosts), and space- and time-translations. The ELT developed below have the remarkable feature that all the parameters of the group fall together in a simple and unified way in a type of overall single rotation, in contrast to the situation in special relativity where the space-rotations and boosts come together but the translations stand rather apart. The ELT, then, are a species of extension and unification of the transformations of ordinary special relativity (but are not to be confounded with general-relativistic transformations of gravitational theory).

The projective transformations have a long history in geometry, and to a degree in physics, in the works of Cartan, Hlavaty, Hoffmann, Kaluza, Klein, Schouten, Veblen, Weyl, and others. References are found in Schouten's book (1). The mathematical groundwork for physical application appears to have been particularly developed by Veblen (2). But, at least in physical context, the projective-geometrical considerations have seemed excessively general, and the physical interpretations opaque at best. The present discussion aims toward mathematical simplicity showing physical connections plainly.

Extended inertial frames

The idea for extending inertial frames is simply this: in OIF (say in one space dimension) a free particle is described by that most important of all physical motions $x = x_0 + vt$. Now if x, t are written as fractional-linear forms in new space-time variables x', t', say $x = L_1/L$ and $t = L_0/L$, with each of L_1, L_0, L being linear (and generally inhomogeneous) functions of x' and t', then it is clear that, owing to L being a common denominator, x' is again a linear function of t'. The fractional-linear transformation preserves the motion as that for a free particle; and the x', t' frame, an EIF, has to be counted as a genuine inertial frame; for the sole and sufficient criterion thereto is exactly the preservation of uniform straight-line motion.

A peculiarity of the EIF statement of the motion at once shows itself. It is that the EIF-reckoned velocity $v' = dx'/dt'$ depends on the origin x_0 of the OIF motion and not just on the OIF velocity v. This origin-dependence is admittedly strange according to traditional physical thought (based on linear space-time transformations). The point is, though, that preservation-of-linear-motion, on the one hand, and origin-independence-of-velocity, on the other hand, are two separate and distinct propositions. We admit the primacy of the first—it is hard to imagine physics without it—and entertain a dropping of the second (at least a *slight* dropping, as described below). In physical issue is whether indeed the origin-independence-of velocity is physically compulsory, to what degree, and whither the development is carried without it. One perhaps cannot detect how necessary or unnecessary a physical hypothesis is until one tries doing without it and sees what consequences follow.

We now have that the transformation law

$$x_i' = (a_i + A_{i\alpha}x_\alpha)/(1 + \sigma_\alpha x_\alpha) \qquad (i = 1,2,3,0) \qquad [1]$$

is the general one which carries us from OIF Cartesians $r \equiv (x_1, x_2, x_3)$ and time $t \equiv x_0$ to EIF ones (summation on repeated index α from 0 to 3). These general projective or EIF transformations of 24 parameters exhaust those preserving uniform motion [Veblen (3) and, in a stimulating discussion, Fock (4)]. It is easy to verify that the EIF transformations make up a group. Moreover, the *homogeneous* EIF transformations, those with the additive constant terms a_i absent in the numerator, separately form a group. These are simpler than the inhomogeneous type and give the starting point for the extension of special relativity.

Abbreviations: OIF, ordinary inertial frames; EIF, extended inertial frames; OLT, ordinary Lorentz transformations; ELT, extended Lorentz transformations.

* This is paper I in a series.

Let the numerator linear-form $A_{i\alpha}x_\alpha$ in [1] be that for the ordinary homogeneous Lorentz transformation (without spatial rotation), characterized wholly by a triplet of velocity parameters v_1, v_2, v_3, or v. A natural try toward EIF generalization, which still holds to a characterization through only v, is to take $\sigma_\alpha x_\alpha$ in the denominator as $hv\cdot r/cb + kct/b$, where b is a new universal length, and h and k are dimensionless constants depending on v^2/c^2 at most. The question is whether two such transformations in succession (the first specified by v, the second by v') can give a composite of the same type, by right choice of h and k. The right composition in fact requires, first, the usual Lorentz velocity-composition rule

$$\gamma''v'' = \gamma'\gamma v + \gamma' v'' \cdot \varphi_v, \quad \gamma'' = \gamma'\gamma(1 + v'\cdot v/c^2)$$

$$(\gamma \equiv (1 - v^2/c^2)^{-1/2}; \quad \varphi_v \equiv 1 + \beta vv; \quad \beta \equiv (\gamma - 1)/v^2),$$

(and also, of course, the usual dyadic composition rule that brings in the resultant $\varphi_{v''}$ compounded with a space-rotation), as well as

$$h''v'' = (h + k'\gamma)v + h'v'\cdot\varphi_v,$$
$$k'' = k + k'\gamma + h'\gamma v\cdot v'/c^2,$$

It is at once evident that the unique choice $h = \gamma$ and $k = \gamma - 1$ solves the latter pair, making it identical with the former pair. *Our result is then that*

$$r' = \frac{\varphi_v\cdot r + \gamma vt}{1 + \gamma v\cdot r/cb + (\gamma - 1)ct/b}$$
$$t' = \frac{\gamma(t + v\cdot r/c^2)}{1 + \gamma v\cdot r/cb + (\gamma - 1)ct/b} \qquad [2]$$

are the homogeneous ELT, *generalizing the homogeneous pure* OLT, *forming a group, and reducing to* OLT *for* $b \to \infty$.

The inverse transformation is obtained, as with OLT, by interchanging r' and r, and t' and t, and replacing v by $-v$. The velocity parameter v is however no longer the actual relative velocity of two frames. For the transformation law for velocities, which may readily be written out, shows that dr'/dt' is not fixed but depends explicitly on the world-point location r',t' (or its image r,t from [2]) as well as v and dr/dt. There simply isn't any relative OIF-EIF velocity, as the space of EIF is not in uniform motion overall with respect to that of OIF. It is a bit like a fluid moving nonuniformly, not *en bloc*, with respect to the OIF fluid, and conversely. The positions of OIF and EIF with respect to one another are, as they have to be, symmetrical. Yet the r',t' frame is an inertial frame: a free particle is a free particle for all: the computation of $d^2r'/(dt')^2$ shows naturally that it is proportional to $d^2r/(dt)^2$, and both vanish if one vanishes.

The EIF particle velocity dr'/dt' can naturally assume any value at all, unrestricted by the "light" velocity. What, then, is the number c to mean? Mostly, for the moment, that it is a universal scale factor for velocity, as b is now a universal scale factor for length. The very meaning of "light", i.e., the structure of electrodynamics, is itself here under question or extension. If the ELT have some standing, then electrodynamics as we know it is an approximate statement in need of revision so that it is ELT covariant rather than OLT covariant. So also for gravitation which hinges, by the principle of equivalence, on the world being elementally flat by free fall into an OIF where OLT reigns (in any sufficiently small region where gravity is operating). With ELT as the transformation rule connecting inertial frames, gravitation as well as electrodynamics and other physical statements would have to be informed with both c and b from the outset. Note too that gravitation could enter into inertia in a novel way in view of the fact that b could as well be replaced by $b^2/[G_0\Sigma m/c^2]$ where G_0 is the universal gravitational constant, and where Σm refers to all the masses in a complete self-contained system (this to ensure b's universality). One must suppose of course that b is perhaps of cosmic size or greater, so that if r,ct,r',ct' cover cosmically modest spans, the b corrections to physical statements are very small.

Notice that for particles passing through the common origin of OIF and EIF the motion $r = Vt$ implies $r' = V't'$ with V' composed by the ordinary Lorentz velocity composition rule, and without any dependence on world-point location. In particular, if $|V| = c$ then $|V'| = c$. This much of special relativity holds; "light" from the common origin sparkles invariantly in OIF and EIF. One sees also that the passage to the Galilean limit by way of $b \to \infty$ first and then $c \to \infty$, does not go in two steps if these limits are reversed in order; because if c is let run to infinity first, the ELT goes to the Galilean transformation directly. That is, there is no scheme of EIF in the purely Newtonian domain.

Let us briefly consider the elementary geometric meaning of ELT. A very simple intermediate transformation holds the key to this. Transform both r and t, and r' and t', by

$$\bar{r} = \frac{r}{1 - ct/b}, \quad \bar{t} = \frac{t}{1 - ct/b}; \quad \bar{r}' = \frac{r'}{1 - ct'/b},$$
$$\bar{t}' = \frac{t'}{1 - ct'/b}.$$

These double homographies bring ELT to *linear* Lorentz-like form,

$$\bar{r}' = \varphi_v\cdot\bar{r} + \gamma v\bar{t}, \quad \bar{t}' = \gamma(\bar{t} + v\cdot\bar{r}/c^2), \qquad [3]$$

as a brief calculation shows. It is easily shown that, working in one space dimension for simplicity, as will be discussed more fully elsewhere, the first homographic pair expresses the projective relation between points in suitable intersecting x,t and $\bar{x},\bar{t}$ planes, by means of rays emanating from a suitable center of projection external to both planes and piercing them.

The homographic space and time variables also illuminate several physical points. First, $(d\bar{t})^2 - (d\bar{r})^2/c^2 \equiv (d\bar{\tau})^2$ is the usual quadratic differential invariant coming from [3], and so we have as a differential invariant for homogeneous ELT, writing out $d\bar{r}(r,t)$ and $d\bar{t}(r,t)$,

$$\{(dt)^2 - [(1 - ct/b)dr + rcdt/b]^2/c^2\}/(1 - ct/b)^4.$$

In particular we find the free-particle action $\bar{L}d\bar{\tau} = -mc^2d\bar{\tau}$ belonging to ELT. This is readily translated into $L(r,t,dr/dt)dt$, which checks out directly to be invariant under infinitesimal ELT and to give the Euler-Lagrange equations of motion $d^2r/(dt)^2 = 0$. The transition to the Hamiltonian goes by the ordinary steps, with simplifying canonical transformation which brings into view the familiar types of conservation laws, to be discussed elsewhere.

Next we have that the fundamental invariant wave operator coming from [3] is $\partial^2/(\partial\bar{r})^2 - \partial^2/c^2(\partial\bar{t})^2$, and this provides on calculation the ELT invariant wave operator

$$\boxdot^2 = (1 - ct/b)^2\left\{\square^2 - \frac{1}{b^2}x_\alpha x_\beta\frac{\partial^2}{\partial x_\alpha\partial x_\beta} + \frac{2x_\alpha}{cb}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial t} - \frac{2x_\alpha}{b^2}\frac{\partial}{\partial x_\alpha} + \frac{2}{cb}\frac{\partial}{\partial t}\right\},$$

where $\square^2$ is the usual D'Alembertian $\partial^2/(\partial r)^2 - \partial^2/c^2(\partial t)^2$. The EIF D'Alembertian $\boxdot^2$ contains first as well as second derivatives, and is both rotationally and space-reversal invariant. But it is not time-reversal invariant, because (apart from the first factor) of the last term $(2/cb)\partial/\partial t$. Accordingly, *any wave* *theory built around* EIF *must be expected to have to give up time-reversal invariance, at least to some slight degree.* A plane wave in $\bar{r},\bar{t}$ of the usual type $\exp i(k\cdot\bar{r} - w\bar{t})$ easily translates into r,t language, where it becomes evident that the EIF phase-velocity is c with an additive term proportional to $c/|k|b$.

The general ELT

A better view of the homogeneous ELT is obtained by going to homogeneous coordinates $r = R/U$, $t = T/U$, $r' = R'/U'$, $t' = T'/U'$, which allows [2] to be written as a homogeneous linear transformation in R,T,U variables in typical projective-geometric fashion. Also a Minkowskian introduction of $icT \equiv X_4$ as well as $ibU = X_5$ is in place, bringing the statement of homogeneous ELT to

$$\begin{bmatrix} R' \\ X_4' \\ X_5' \end{bmatrix} = \left(\begin{array}{cc|c} \multicolumn{2}{c}{\psi(v)} & \\ \varphi_v & -i\gamma v & 0 \\ i\gamma v & \gamma & 0 \\ \hline i\gamma v & \gamma - 1 & 1 \end{array}\right)\begin{bmatrix} R \\ X_4 \\ X_5 \end{bmatrix}, \qquad [4]$$

or $X' = \Phi X$ for short. Here v is now abbreviation for v/c, and an arbitrary non-zero factor that may in general multiply the left-hand side has been set equal to unity for convenience. This split notation is useful; a moment's consideration always shows unambiguously how the dyadic, vector, or scalar elements of the split matrix are to be multiplied into the elements of a like matrix or into those of a split 5-fold vector like (R,X_4,X_5). In the 4×4 box labeled $\psi(v)$ we have a conventional Lorentz (hyperbolic) rotation.

Our task is to understand how to replace the zeroes in the fifth column with finite elements so that we get general and *inhomogeneous* fractional-linear forms when we return to x_α' in terms of x_α. These will be the transformations that for EIF will correspond to the full inhomogeneous group for OIF. It will be clear that the fifth coordinate U or X_5 is simply and solely a convenient device for handling the fractional-linear forms. It is not to be thought of as any sort of "new" coordinate on the same standing as the fundamental space-time variables x_α; working formally in the 5-dimensional space can have an appeal of elegance, but the risk is an order of abstractness that can negate physical comprehensibility. The fractional-linear forms in the primitive space-time variables x_α remain the conveyances of physics.

By subtracting X_5' from X_4' we find that $X_4' - X_5' = X_4 - X_5$. At the same time $R'^2 + X_4'^2 = R^2 + X_4^2$. The nature of the homogeneous ELT of [4] is then that the quadratic form

$$Q \equiv X_1^2 + X_2^2 + X_3^2 + X_4^2 + f^2(X_5 - X_4)^2$$

is invariant, where f^2 is an arbitrary pure numeric, whose value (either positive or negative) cannot be settled mathematically but only, if at all, physically. Plainly, a 5-dimensional rotation is hiding here. A simple transformation brings it out directly, namely, Φ can be written as a similarity transform of a rotation: $\Phi \equiv F^{-1}R_LF$, or

$$\Phi = \begin{pmatrix} I & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & f^{-1} \end{pmatrix}\begin{pmatrix} \varphi_v & -i\gamma v & 0 \\ i\gamma v & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}\left(\begin{array}{c|cc} I & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & -f & f \end{array}\right),$$

I being the unit dyadic. This says that $FX' = R_L(FX)$ or that $(FX)\cdot(FX)$ is invariant, and this it is that is Q above.

The way to the general ELT is now clear. We take it as a *fundamental hypothesis, that the general ELT is defined through* $X' = F^{-1}\mathcal{R}_5FX$ *where* $\mathcal{R}_5$ *is a suitable general 5-dimensional rotation.* Due to orthonormality restrictions, an $\mathcal{R}_5$ can have up to 10 free parameters, naturally right to account for x_α' being connected to x_α through a 10-parameter group.

It is evident that the 10 parameters are all entwined together under the one $\mathcal{R}_5$, in contrast to the group of inhomogeneous OLT where the translations are, so to say, tacked on to the six others and not integral with them. In particular, one sees that in the group of ELT the space- and time-translational features are a species of rotation. Dynamical conservation laws associated with the ELT group are all expected to be of the nature of angular-momentum (in the wide five-sense)-conservation laws.

It turns out to be useful to bring in another similarity transformation, one involving a pure X_4,X_5 rotation through $\pi/2$. Let us place $\mathcal{R}_5 = S^{-1}\mathcal{R}_5{}^*S$. ($\mathcal{R}_5{}^*$ being another 5-rotation) with S like F above, except that the 2×2 square box in the lower right corner is replaced by the 2×2 matrix having $[0, -1]$ in the first row and $[1, 0]$ in the second row. Then, the R_L above is $S^{-1}R_{5L}{}^*S$ with $R_{5L}{}^*$ appropriately similar to R_L. Now just as ψ above with three parameters v characterizes the OLT group (aside from spatial rotation), we can characterize a correspondingly general type of $\mathcal{R}_5{}^*$ by four parameters $(u_1, u_2, u_3; u_0) \equiv (u; u_0)$ in a matrix of type

$$R_5{}^*(u) = \begin{pmatrix} \varphi_u & Buu_0 & -iGu \\ Buu_0 & 1 + Bu_0{}^2 & iGu_0 \\ iGu & -iGu_0 & G \end{pmatrix}$$

with $\varphi_u \equiv 1 + Buu$. A moment's computation shows that this is indeed a rotation when $G = (1 - u_\alpha u_\alpha)^{-1/2}$ and $B = (G - 1)/u_\alpha u_\alpha$. It is merely a step beyond ψ, embracing five instead of four dimensions. If $(u;u_0)$ is $(v;0)$ we just recover $\mathcal{R}_{5L}{}^*$. If instead we take $(u;u_0) = (v;i\eta)$ we have a suitable initial generalization of $\mathcal{R}_{5L}{}^*$. The v is abbreviation for the dimensionless v/c. Now we let η be abbreviation for the dimensionless η/b, where η is a new length parameter on the same footing as v, so that for instance G is $(1 - v^2/c^2 + \eta^2/b^2)^{-1/2}$; note that $|v/c|$ is no longer restricted to be <1, owing to the η^2/b^2.

Recall finally how the ordinary homogeneous Lorentz transformation is generalized to account for both space-rotations and boosts. It is done by combining a 3-rotation premultiplied into a pure ψ-type Lorentz transformation. On this same path we have that the general-$\mathcal{R}_5{}^*$ is to be gotten by premultiplying the $\mathcal{R}_5{}^*(u)$ above by a general rotation in X_1, X_2, X_3, X_4 space. The latter, in turn, is of type (3-rotation)$\cdot$(pure 4-rotation of ψ-structure but with new parameters of its own). We have for the final general 10-parameter 5-space rotation,* call it $\{\mathcal{R}_5{}^*\}$, the product $\mathcal{R}_3\mathcal{R}_4(\xi)\mathcal{R}_5{}^*(u)$. The rotation $\mathcal{R}_3$ is a 5×5 matrix with an ordinary 3×3 space-rotation in the upper left corner (characterized by a set of Euler angles in 3-space), unity in the (4,4) and (5,5) places, and zeroes elsewhere. The $\mathcal{R}_4(\xi)$ is a 5×5 matrix with $\psi(\xi)$ in the upper left 4×4 corner, unity in the (5,5) place, and zeroes elsewhere. The $\psi(\xi)$ is exactly like $\psi(v)$ with, however, ξ standing as abbreviation for (3-vector space displacement $\xi)/b$; it is a kind of Lorentz rotation, but in ξ/b rather than v/c. The general $\{\mathcal{R}_5{}^*\}$ is built so as to recover the homogeneous ELT for $\mathcal{R}_3 = 1$ and $\xi,\eta = 0$, and so that when r',t' are expressed as fractional-linear forms in r,t, and the limit $b \rightarrow \infty$ taken, the *full inhomogeneous ordinary Lorentz transformations are obtained.* The group property of all ELT is a consequence of that of the $\{\mathcal{R}_5{}^*\}$.

On total we have for the general Φ the combination $F^{-1}S^{-1}\{\mathcal{R}_5{}^*\}SF$. The matrix multiplication is readily done, as is the return to r',t' and r,t variables. This is a little cumbersome to write out in the present place; suffice it to mention the $b \rightarrow \infty$ limit,

$$r' = \varphi_v\cdot r + \gamma vt + f\eta\beta cv - f\xi$$

$$t' = \gamma(t + v\cdot r/c^2) - f\gamma\eta/c,$$

with *non*-abbreviated quantities here, ξ, η = (length), and v = (velocity). The $\mathcal{R}_3$ has been taken as unity for simplicity. This contains, as sought and necessary, arbitrary inhomogeneous additions to the homogeneous OLT.

Our result is that the Law of Inertia, in and by itself, admits an extension of the Lorentz group; the parameters of the extended group are all bound together in rotations in the space of homogeneous coordinates. The transformations of general type contain the velocity-scaling constant c, the new length-scaling constant b, and a dimensionless, so far arbitrary, numeric f.

The possible restructuring of electrodynamics, and as well the restructuring of other physical statements, and the possible physical consequences that may be experimentally detectable, are expected to be discussed on another occasion.

Note Added in Proof. The fuller exposition of inhomogeneous ELT has been submitted to the *Journal of Mathematical Physics*, including the Lie Group algebra; differential geometry (generally that for a Finsler rather than a Riemann space); free-particle dynamics (requiring an eight-branched Hamiltonian); and preliminary electrodynamics (requiring both charge and a second electromagnetic parameter).

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