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Mathematics

Nonlinear wave equations and constrained harmonic motion

(dynamical systems/isospectral deformations)

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ABSTRACT The study of the Korteweg–deVries, nonlinear Schrödinger, Sine–Gordon, and Toda lattice equations is simply the study of constrained oscillators. This is likely to be true for any nonlinear wave equation associated with a second-order linear problem.

In this note we wish to announce some results concerning nonlinear wave equations whose initial value problem can be solved by the inverse scattering method. Briefly, we have found that the study of a number of equations describing the isospectral deformation of second-order ordinary difference/differential operators can be reduced to the study of a system of free oscillators constrained to an intersection of quadrics in phase space. Moreover, this constrained motion is integrable. All that changes from one wave equation to another is the variety of constraint. These results are in analogy with the linear theory in which the study of linear wave equations with constant coefficients is reduced via the Fourier transform to the study of harmonic oscillators, but now with linear constraints. Here we illustrate the method for the Korteweg–deVries equation, already treated by Moser and Trubowitz (see ref. 1), and sketch our new results for the nonlinear Schrödinger equation, the Sine–Gordon equation in laboratory coordinates, and the Toda lattice. We have no doubt that the technique can be directly applied to the continuous Heisenberg spin chain (2), the generalized Sine–Gordon equation (3), and the classical Thirring model (4).

The Korteweg–deVries equation

Let $q(x)$ be a real-valued function on the line, vanishing as $x \to \pm\infty$ and let f_k, g_k be the solutions of

$$-y'' + q(x)y = k^2 y \qquad [1]_k$$

with $f_k \underset{x\to+\infty}{\sim} e^{ikx}$ and $g_k \underset{x\to-\infty}{\sim} e^{-ikx}$. We have

$$g_k \underset{x\to+\infty}{\sim} \frac{1}{T(k)}\left(e^{-ikx} + R(k)e^{ikx}\right),$$

where $R(k)$ is the reflection coefficient and $T(k)$ is the transmission coefficient for q. For simplicity, we will assume that $-d^2/dx^2 + q$ has no bound states or resonances.

In ref. 5 it was shown that

$$1 = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{R(\ell)}{\ell} f_\ell^2(x)\, d\ell. \qquad [2]$$

By differentiating once we obtain

$$0 = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{R(\ell)}{\ell} f_\ell(x) f'_\ell(x)\, d\ell, \qquad [3]$$

and differentiating again,

$$q(x) = -\frac{i}{\pi}\int_{-\infty}^{\infty} \frac{R(\ell)}{\ell} f_\ell'^2(x) + \frac{i}{\pi}\int_{-\infty}^{\infty} \ell R(\ell) f_\ell^2(x)\, d\ell. \qquad [4]$$

Now consider the map

$$q(x) \to \left(x_k = \sqrt{\frac{i}{\pi}\frac{R(k)}{k}}\, f_k(0),\quad y_k = \sqrt{\frac{i}{\pi}\frac{R(k)}{k}}\, f'_k(0);\quad -\infty < k < \infty\right).$$

Under this change of variables translation, $\partial q/\partial t = \partial q/\partial x$, becomes

$$\dot{x}_k = y_k,\ \dot{y}_k = -k^2 x_k + \left[-\int_{-\infty}^{\infty} y_\ell^2\, d\ell + \int_{-\infty}^{\infty} \ell^2 x_\ell^2\, d\ell\right] x_k, \qquad [5]_k$$

which is precisely the Schrödinger equation $[1]_k$ with q eliminated through [4]. Also, [2] and [3] become

$$1 = \int_{-\infty}^{\infty} x_k^2\, dk \qquad [6]$$

and

$$0 = \int_{-\infty}^{\infty} x_k y_k\, dk. \qquad [7]$$

We now give an interpretation of these equations. Let

$$\ddot{x}_k + k^2 x_k = 0, \qquad -\infty < k < \infty,$$

be a system of free harmonic oscillators with Hamiltonian $H_0 = \frac{1}{2}\int_{-\infty}^{\infty} y_k^2 + k^2 x_k^2\, dk$ and let X be the variety defined by

$$\phi_1 \equiv \int_{-\infty}^{\infty} x_k^2\, dk - 1 = 0$$

$$\phi_2 \equiv \int_{-\infty}^{\infty} x_k y_k\, dk = 0.$$

Now constrain the free motion to X by writing $H = H_0 + \alpha_1\phi_1 + \alpha_2\phi_2$ and imposing the conditions

$$\{\phi_1, H\} = 0,$$
$$\{\phi_2, H\} = 0$$

on X.

We find

$$\alpha_2 = 0,$$

$$\alpha_1 = \frac{1}{2}\int_{-\infty}^{\infty} y_k^2 - k^2 x_k^2\, dk,$$

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so that the equations of constrained motion are

$$\frac{d}{dt}x_k = \{x_k,H\} = y_k$$

$$\frac{d}{dt}y_k = \{y_k,H\} = -k^2x_k + \left[-\int_{-\infty}^{\infty} y_\ell^2 d\ell + \int_{-\infty}^{\infty} \ell^2 x_\ell^2\, d\ell\right] x_k.$$

But this is $[5]_k$.

Furthermore, the motion is integrable. To see this set

$$A_k = x_k^2 + \int_{-\infty}^{\infty} \frac{(x_k y_\ell - x_\ell y_k)^2}{k^2 - \ell^2} d\ell,$$

$-\infty < k < \infty$.

A direct calculation shows that

$$\{A_k,A_\ell\}_X = 0, \qquad -\infty < k,\ell < \infty,$$

$$\{A_k,H\}_X = 0,$$

and

$$\int_{-\infty}^{\infty} k^2 A_k dk = 2H_0 = 2H \ (\mathrm{mod}\ X).$$

Here, $\{\cdot,\cdot\}_X$ is the Poisson bracket on X.

The whole point is that under the same change of variables the evolution of q given by the Korteweg–deVries equation, $\partial q/\partial t = 3qq_x - 1/2\, q_{xxx}$, becomes the motion on X generated by the Hamiltonian $H_{KdV} = \int_{-\infty}^{\infty} k^4 A_k\, dk$. Clearly, $\{H_{KdV},H_0\}_X = 0$, so that the study of the Korteweg–deVries equation is reduced to the study of constrained harmonic motion.

The nonlinear Schrödinger equation

Let q, r be complex valued functions vanishing at $\pm\infty$ and let ψ_+, ψ_- be solutions of

$$\begin{aligned} v_1' + ikv_1 &= qv_2 \\ v_2' - ikv_2 &= rv_1 \end{aligned} \qquad [8]_k$$

with

$$\psi_+(x,k) = \begin{pmatrix}\psi_{+1}(x,k)\\ \psi_{+2}(x,k)\end{pmatrix} \underset{x\to+\infty}{\sim} \begin{pmatrix}0\\1\end{pmatrix} e^{ikx}$$

and

$$\psi_-(x,k) = \begin{pmatrix}\psi_{-1}(x,k)\\ \psi_{-2}(x,k)\end{pmatrix} \underset{x\to+\infty}{\sim} \begin{pmatrix}1\\0\end{pmatrix} e^{-ikx}.$$

We have

$$\psi_+(x,k) \underset{x\to-\infty}{\sim} -a_+(k)\begin{pmatrix}0\\-1\end{pmatrix} e^{ikx} + b_-(k)\begin{pmatrix}1\\0\end{pmatrix} e^{-ikx}$$

and

$$\psi_-(x,k) \underset{x\to-\infty}{\sim} a_-(k)\begin{pmatrix}1\\0\end{pmatrix} e^{-ikx} + b_+(k)\begin{pmatrix}0\\-1\end{pmatrix} e^{ikx}.$$

Set[§] $r_+(k) = \dfrac{b_+(k)}{a_+(k)}$, $r_-(k) = \dfrac{b_-(k)}{a_-(k)}$. Then

$$q(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} r_+(k)\psi_{+1}^2(x,k) + r_-(k)\psi_{-1}^2(x,k)\, dk \quad [9]$$

and

$$r(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} r_+(k)\psi_{+2}^2(x,k) + r_-(k)\psi_{-2}^2(x,k)\, dk. \quad [10]$$

Now, under the change of variables

$$(q,r) \to [x_{+k}, y_{+k}, x_{-k}, y_{-k};\ -\infty < k < \infty],$$

where[¶]

$$x_{+k} = \sqrt{\frac{r_+(k)}{\pi k}}\,\psi_{+1}(0,k),\ x_{-k} = \sqrt{\frac{r_-(k)}{\pi k}}\,\psi_{-1}(0,k) \qquad (k \neq 0)$$

$$y_{+k} = \sqrt{\frac{r_+(k)}{\pi k}}\,\psi_{+2}(0,k),\ y_{-k} = \sqrt{\frac{r_-(k)}{\pi k}}\,\psi_{-2}(0,k),$$

translation becomes

$$\frac{d}{dt}x_{\pm k} = -ikx_{\pm k} - \left[\int_{-\infty}^{\infty} \ell(x_{+\ell}^2 + x_{-\ell}^2)\, d\ell\right] y_{\pm k}$$

$$\frac{d}{dt}y_{\pm k} = iky_{\pm k} + \left[\int_{-\infty}^{\infty} \ell(y_{+\ell}^2 + y_{-\ell}^2)\, d\ell\right] x_{\pm k}. \qquad [11]_k$$

As before, system $[11]_k$ is obtained by coupling $[8]_k$ together through [9] and [10].

Our first result is that, as in the Korteweg–deVries equation, $[11]_k$ can be realized by constraining the harmonic oscillators

$$H_0 = -\int_{-\infty}^{\infty} ik[x_{+k}y_{+k} + x_{-k}y_{-k}]\, dk$$

to

$$x_{+0}^2 + x_{-0}^2 + \int_{-\infty}^{\infty} x_{+k}^2 + x_{-k}^2\, dk = 0,$$

$$y_{+0}^2 + y_{-0}^2 + \int_{-\infty}^{\infty} y_{+k}^2 + y_{-k}^2\, dk = 0$$

in the Poisson structure $\{x_{+k},y_{+\ell}\} = \{x_{-k},y_{-\ell}\} = \delta_{k\ell}$, $\{x_{+k},x_{-\ell}\} = \{y_{+k},y_{-\ell}\} = \{x_{+k},y_{-\ell}\} = \{x_{-k},y_{+\ell}\} = 0$. The resulting system is larger than $[11]_k$, which is obtained by setting $x_{+0}y_{+0} + x_{-0}y_{-0} + \int_{-\infty}^{\infty} x_{+k}y_{+k} + x_{-k}y_{-k}\, dk$, a conserved quantity, equal to $-i$.

Define

$$B_{\pm k} \equiv \frac{(x_{\pm k}y_{+0} - x_{+0}y_{\pm k})^2 + (x_{\pm k}y_{-0} - x_{-0}y_{\pm k})^2}{k} + \int_{-\infty}^{\infty} \frac{(x_{\pm k}y_{+\ell} - x_{+\ell}y_{\pm k})^2 + (x_{\pm k}y_{-\ell} - x_{-\ell}y_{\pm k})^2}{k-\ell}\, d\ell.$$

The $B_{\pm k}$s Poisson commute and are conserved by the constrained system so that the motion is integrable. Finally, the nonlinear Schrödinger equation (6, 7), $i\,\partial q/\partial t + \partial^2/\partial x^2 q + |q|^2 q = 0$, is identified as the flow generated by the Hamiltonian $H_{nS} = \int_{-\infty}^{\infty} k^3(B_k + B_{-k})\, dk$ along trajectories with

$$x_{-k} = y_{+k}^*,\ y_{-k} = -x_{+k}^*.$$

The Sine–Gordon Equation

Let $u(t,x)$ be a solution of the Sine–Gordon equation (8, 9) $u_{tt} - u_{xx} + \sin u = 0$ with $u(t,x) \to 0$ as $x \to \pm\infty$. Set $w_+ = 1/4\,(u_x + u_t)$, $w_- = 1/4\,(u_x - u_t)$. Let ψ_+, ϕ be the solutions of

$$\begin{aligned} v_1' &= \left(-\frac{ik}{2} + \frac{i}{8k}\cos u\right)v_1 + \left(-\frac{w_+}{4} + \frac{i}{8k}\sin u\right)v_2 \\ v_2' &= \left(\frac{w_+}{4} + \frac{i}{8k}\sin u\right)v_1 + \left(\frac{ik}{2} - \frac{1}{8k}\cos u\right)v_2 \end{aligned} \qquad [12]_k$$

with

$$\psi_+(x,k) = \begin{pmatrix}\psi_{+1}(x,k)\\ \psi_{+2}(x,k)\end{pmatrix} \underset{x\to\infty}{\sim} \begin{pmatrix}0\\1\end{pmatrix} e^{i(\frac{k}{2} - \frac{1}{8k})x}$$

§ For simplicity, we are assuming that both a_+, a_- are root free.

¶ x_{+0}, x_{-0}, y_{+0}, and y_{-0} are defined in a somewhat different manner. We omit the precise definition because it requires a technical digression.

and

$$\phi(x,k) = \begin{pmatrix}\phi_1(x,k)\\ \phi_2(x,k)\end{pmatrix} \underset{x\to-\infty}{\sim} \begin{pmatrix}1\\0\end{pmatrix} e^{-i(\frac{k}{2}-\frac{1}{8k})x}.$$

We have

$$\phi(x,k) \underset{x\to+\infty}{\sim} a(k)\begin{pmatrix}1\\0\end{pmatrix} e^{-i(\frac{k}{2}-\frac{1}{8k})x} + b(k)\begin{pmatrix}0\\1\end{pmatrix} e^{i(\frac{k}{2}-\frac{1}{8k})x}.$$

Set $r_k = b(k)/a(k)$.‖

Now

$$w_+ = \frac{2}{\pi}\int_{-\infty}^{\infty} r_k(\psi_{+1}^2 + \psi_{+2}^2)\,dk \qquad [13]$$

$$\sin u = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{r_k}{k}(\psi_{+1}^2 - \psi_{+2}^2)\,dk \qquad [14]$$

$$\cos u - 1 = -\frac{2i}{\pi}\int_{-\infty}^{\infty} \frac{r_k}{k}\psi_{+1}\psi_{+2}\,dk \qquad [15]$$

$$w_- = -\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{r_k}{k^2}(\psi_{+1}^2 + \psi_{+2}^2)\,dk. \qquad [16]$$

Define $\psi_-(x,k) = \exp\{i(\begin{smallmatrix}0 & -i\\ i & 0\end{smallmatrix})u/2\}\psi_+(x,k)$. Then ψ_- is a solution of

$$v_1' = \left(\frac{i}{8k} - \frac{ik}{2}\cos u\right)v_1 + \left(\frac{w_-}{4} + \frac{ik}{2}\sin u\right)v_2$$

$$[17]_k$$

$$v_2' = \left(-\frac{w_-}{4} + \frac{ik}{2}\sin u\right)v_1 + \left(-\frac{i}{8k} + \frac{ik}{2}\cos u\right)v_2.$$

Of course, the functions $w_\pm$, $\sin u$, and $\cos u - 1$ can also be expressed in terms of ψ_-.

Under the change of variables**

$$x_k = \sqrt{\frac{r_k}{\pi k \Lambda_k}}\,\psi_{+1}(0,k),\ u_k = \sqrt{\frac{r_k}{\pi k \Lambda_k}}\,\psi_{-1}(0,k)$$

$$(k \neq 0)$$

$$y_k = \sqrt{\frac{r_k}{\pi k \Lambda_k}}\,\psi_{+2}(0,k),\ v_k = \sqrt{\frac{r_k}{\pi k \Lambda_k}}\,\psi_{-2}(0,k),$$

translation is realized by constraining the harmonic oscillators

$$H_0 = -\frac{i}{2}\int_{-\infty}^{\infty} k\,x_k y_k\,dk + \frac{i}{8}\int_{-\infty}^{\infty}\frac{1}{k}u_k v_k\,dk$$

to the variety X given by

$$\int_{-\infty}^{\infty} x_k^2 + y_k^2\,dk = 0$$

$$\int_{-\infty}^{\infty} u_k^2 + v_k^2\,dk = 0$$

$$\int_{-\infty}^{\infty} k(x_k^2 - y_k^2)\,dk = 0$$

$$\int_{-\infty}^{\infty} \frac{1}{k}(u_k^2 - v_k^2)\,dk = 0$$

$$\int_{-\infty}^{\infty}\left(\frac{1}{4k}x_k y_k - k\,u_k v_k\right)dk = 0$$

$$\int_{-\infty}^{\infty}\frac{1}{4k}\left(x_k^2 - y_k^2\right) + k\left(u_k^2 - v_k^2\right)dk = 0,$$

with the restriction (which is preserved in time)

$$u_k = \cos\frac{u}{2}x_k + \sin\frac{u}{2}y_k$$

$$v_k = -\sin\frac{u}{2}x_k + \cos\frac{u}{2}y_k.$$

This constrained motion is again integrable. The conserved quantities are

$$c_{1k} = \int_{-\infty}^{\infty}\frac{k+\ell}{k-\ell}(x_k y_\ell - x_\ell y_k)^2 + \frac{k-\ell}{k+\ell}(x_k x_\ell + y_k y_\ell)^2\,d\ell$$

$$(-\infty < k < \infty)$$

$$c_{2k} = \int_{-\infty}^{\infty}\frac{k+\ell}{k-\ell}(u_k v_\ell - u_\ell v_k)^2 + \frac{k-\ell}{k+\ell}(u_k u_\ell + v_k v_\ell)^2\,d\ell.$$

Finally, under the same change of variables the Sine–Gordon motion is obtained by constraining

$$H_{sG} = \frac{i}{2}\int_{-\infty}^{\infty}\frac{1}{k}x_k y_k\,dk - \frac{i}{8}\int_{-\infty}^{\infty}k\,u_k v_k\,dk$$

to X and imposing the above restrictions.

The Toda lattice

Let $(a,b) \in (R^+)^n \oplus R^n$ and let L be the periodic difference operator defined by

$$Lf = [a_{\ell-1}f(\ell-1) + b_\ell f(\ell) + a_\ell f(\ell+1),\ \ell \in Z],$$

with $a_{\ell+n} = a_\ell$, $b_{\ell+n} = b_\ell$, and $f(\ell+n) = f(\ell)$ for all ℓ. For simplicity we will assume that the spectrum of L is simple.

Now let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of L and let $f_1, \ldots, f_n$ be the corresponding normalized eigenfunctions. We have

$$b_\ell = \sum_{i=1}^n \lambda_i f_i^2(\ell) \qquad [18]$$

$$a_\ell = \sum_{i=1}^n \lambda_i f_i(\ell) f_i(\ell+1) \qquad [19]$$

for all ℓ. Set $x_\ell = \sqrt{a_1}f_\ell(1)$, $y_\ell = \sqrt{a_1}f_\ell(2)$, $\ell = 1, \ldots, n$. Under the change of variables

$$(a,b) \to (x_1,y_1,\ldots,x_n,y_n),$$

the periodic Toda differential equations

$$\frac{d}{dt}a_\ell = a_\ell(b_{\ell+1} - b_\ell)$$

$$\ell = 1,\ldots,n, \qquad [20]$$

$$\frac{d}{dt}b_\ell = 2(a_\ell^2 - a_{\ell-1}^2),$$

with $a_0 = a_n$, $b_{n+1} = b_1$, become

$$\frac{d}{dt}x_\ell = \left[\frac{\sum_{i=1}^n \lambda_i(x_i^2 + y_i^2)}{\sum_{i=1}^n x_i^2 + y_i^2} - \lambda_\ell\right]x_\ell + \left[\sum_{i=1}^n x_i^2 + y_i^2\right]y_\ell$$

$$[21]$$

$$\frac{d}{dt}y_\ell = \left[\lambda_\ell - \frac{\sum_{i=1}^n \lambda_i(x_i^2 + y_i^2)}{\sum_{i=1}^n x_i^2 + y_i^2}\right]y_\ell - \left[\sum_{i=1}^n x_i^2 + y_i^2\right]x_\ell.$$

‖ For simplicity we are assuming that $a(k)$ is root free.

** Again, we omit the precise definition at $k = 0$. Also, $\Lambda_k = k/2 - 1/8k$.

System **21** is obtained by constraining

$$H_0 = -\sum_{i=1}^{n} \lambda_i x_i y_i + \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$$

$$\text{to} \sum_{i=1}^{n} x_i y_i = 0, \sum_{i=1}^{n} y_i^2 - x_i^2 = 0.$$

Moreover, the polynomials

$$D_k = x_k y_k + \sum_{\ell \neq k} \frac{(x_k y_\ell - x_\ell y_k)^2}{\lambda_\ell - \lambda_k}, \qquad 1 \leq k \leq n$$

are Poisson-commuting conserved quantities for the motion.

It is possible to give a similar construction for Toda particles on the line.

The procedure we have outlined provides a unifying scheme for all the nonlinear wave equations associated with a second-order linear problem. Under an appropriate transformation they become different aspects of the same simple system, harmonic oscillators.

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