

inflected. All this fits in very well with the theory of Helmholtz and Lippmann based on the 2nd law of thermodynamics. One may regard the kinetic pressure of the negative ions within the metal as continually in excess of the pressure of the positive ions; so that in like fields of the double layer, the former escape sooner.

THE TRANSFER IN QUANTA OF RADIATION MOMENTUM TO MATTER

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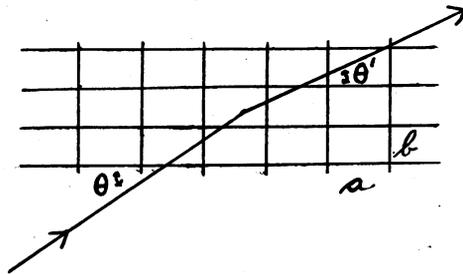
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1. The reflection by a crystal of X-radiation characteristic of the atoms in the crystal itself, which Dr. G. L. Clark and the writer discovered (these PROCEEDINGS, May, 1922 and April, 1923), does not appear to be explainable in a simple manner by the theory of interference of waves. This note describes an attempt to formulate a theory of the reflection of X-rays by crystals, based on quantum ideas without reference to interference laws.

2. The fundamental hypothesis of the theory now presented is that the *momentum* of radiation is transferred to and from matter in quanta, and further, that the laws of the conservation of energy and of momentum apply to these transfers.

3. In order to illustrate the meaning of this hypothesis, let us take a particular example, namely, that of the reflection of an X-ray by a crystal. For the sake of simplicity, let us assume that the axes of the crystal lie



at right angles to each other and take the particular case in which the X-ray strikes the crystal in a direction parallel to one set of principal planes. The problem thus becomes a two-dimensionable one. Suppose

that the distances between parallel planes are a and b , respectively, and let an X-ray of energy $h\nu$ and momentum $h\nu/c$ pass through the crystal at an angle θ with the planes, called x , as represented in the figure. At some point in its path the X-ray may be deflected so as to travel in a direction, making an angle θ' with the x planes. At this point of deflection the X-ray transfers some momentum and energy to the crystal. The mass of the crystal being so large, however, the velocity acquired by it must be very small and, consequently, the energy transferred from the radiation to the crystal may be neglected. We will take up later the case where the X-ray may break up one of the atoms of the crystal, ejecting an electron and, therefore, transferring energy to the atom on that account. Neglecting the amount of energy transferred to the crystal, we may put $h\nu' = h\nu$ and we may consider that the X-ray has the same energy and total momentum after the transfer as before.

The momenta transferred to the crystal in the directions x and y , respectively, are expressed by the following equations:

$$M_x = h\nu/c.(\cos \theta - \cos \theta'), \quad M_y = h\nu/c.(\sin \theta - \sin \theta'). \quad (1)$$

According to the fundamental hypothesis the momentum M_x transferred to the crystal is to be delivered in quanta. We may write, therefore, M_x proportional to the constant h . Now, let us apply to the problem dimensional reasoning. The momentum has the dimensions ml/t and the constant h has the dimensions ml^2/t . In order, therefore, that both sides of our equation may have the same dimensions, we must divide h by a length. Since this is a crystalline phenomenon the only quantity of importance having the dimensions of a length in the direction of the axis of X is the distance a between the crystal planes. Since any whole number, τ , of quanta may be transferred, we will multiply τ into the quotient of h by a and we reach the conclusion that the momentum M_x must be equal to $\tau h/a$ multiplied by some numerical constant. The equation will take its simplest form if we put this constant factor of proportionality equal to unity. We thus get the equation: $M_x = \tau h/a$. Similarly, $M_y = \tau' h/b$. Another way of looking at these equations is to regard them as peculiar applications to the *transfers* of momenta of the quantum equation, $\int p dq = nh$, expressed in generalized coordinates. Substituting for M_x and M_y in equations (1), we get immediately:

$$h\nu/c.(\cos \theta - \cos \theta') = \tau h/a, \quad h\nu/c.(\sin \theta - \sin \theta') = \tau' h/b. \quad (2)$$

4. In the above equations τ and τ' represent whole numbers. If $\tau = \tau' = 0$, $\theta' = \theta$, and the incident radiation quantum passes through the crystal without deflection. It will keep on traveling through the crystal until it reaches a point where either τ or τ' differs from 0. Suppose

that τ' differs from 0 at some point, and that τ remains 0, then, from the first of equations (2) it follows that $\cos \theta' = \cos \theta$. θ' , however, cannot equal θ , for in this case τ' does not equal 0. Consequently, θ' must equal $-\theta$. This means that the X-ray at the point is deflected downward in a direction making an angle with the x -planes equal to the angle of incidence. In other words, the angle of reflection by the x -planes equals the angle of incidence on them.

Substituting $\theta' = -\theta$ in the second of equations (2) we get:

$$2h\nu/c \cdot \sin \theta = \tau'h/b, \quad (3)$$

which represents the relation that must exist between ν and θ in order that τ' may differ from zero. Putting $\nu/c = 1/\lambda$, equation (3) reduces to:

$$\tau'\lambda = 2b \sin \theta.$$

This is Bragg's law of reflection of an X-ray by a crystal.

By similar reasoning, if $\tau' = 0$ and τ differs from 0, we deduce the equation:

$$\tau\lambda = 2a \cos \theta,$$

which represents reflection according to Bragg's law from the y -planes. If both τ and τ' differ from 0, the equation obtained reduces to Bragg's equation representing the reflection from a set of planes other than the principal planes. In the case where the axes of the crystal are not at right angles to each other, we apply the law of the transfer of momenta in quanta to the total component of the radiation momentum in the direction of each axis and we equate this component to $\tau h/a$ where a is the parameter of the crystal along the axis. This gives us equations which reduce to Bragg's equation for the reflection from each set of planes, as in the orthogonal problem.

5. The reasoning by which we have deduced Bragg's law of reflection of X-rays by a crystal cannot be considered a logical demonstration. We may regard the reasoning, however, as a means of suggesting equations to be tested by applying them to experimentally determined facts. We know, for instance, that Bragg's law holds to a very high degree of precision. On the other hand, the same may be said of all physical theories. General fundamental assumptions form the basis of every physical theory and, no matter how rigorously we may deduce from them laws and facts to be tested by experiment, the theories, on account of the hypothetical character of the fundamental principles, should always be regarded purely as methods of suggesting the laws to be experimentally tested.

6. In the above discussion of the reflection of X-rays by crystals nothing has been specifically said about wave-lengths or frequencies of vibration, although in order to conform with usage the energy of the incident radiation quantum was put equal to $h\nu$, and in order to compare the formulas obtained with the usual expressions of them, the wave-length, λ , was introduced. We might just as well have called the energy ϵ and introduced it into the equations.

In deducing more general equations for the defraction of X-rays by a crystal, let us follow this procedure. Let us take the axes of x , y and z to coincide with the axes of the crystal, having respectively, the parameters d_1 , d_2 and d_3 . Denote the direction cosines of the incident X-ray, of energy ϵ , by l , m and n , and those of the deflected ray by l' , m' and n' . Then, the momentum of the ray being ϵ/c , we may write the following equations for the amounts of momentum transferred to the crystal by the ray in the directions of the axes x , y and z , respectively.

$$(1 - l')\epsilon/c = \tau_1 h/d_1, \quad (m - m')\epsilon/c = \tau_2 h/d_2, \quad (n - n')\epsilon/c = \tau_3 h/d_3 \quad (4)$$

where τ_1 , τ_2 and τ_3 are whole numbers. In addition to these three equations we have the usual relations between the direction cosines l , m , and n , and between the direction cosines l' , m' and n' .

7. To compare these equations with the forms often used to describe the defraction of rays by a crystal we may put $\epsilon = hc/\lambda$ where λ denotes some parameter connected with the radiation quantum ϵ , but need not be regarded in our present theory as the distance between waves, as in the wave theory. Equations (4) then become:

$$\tau_1 \lambda = d_1(l - l'), \quad \tau_2 \lambda = d_2(m - m'), \quad \tau_3 \lambda = d_3(n - n'),$$

general equations for the defraction of X-rays by a crystal.

8. In the case of the reflection of rays characteristic of the chemical elements in the crystal, the theory becomes more complicated. Here we must consider that a certain amount of the energy of the incident ray goes to produce the emission of an electron from an atom with a certain velocity and we must take into account the transfer of momentum to the atom and the emitted electron, as well as to the crystal. Perhaps the simplest fundamental assumption to make is that the incident quantum of radiation delivers part of its momentum to the atom and the electron on the one hand and quantizes the rest with the crystal as in the foregoing examples. From this idea we obtain the equations

$$\begin{aligned} l\epsilon/c - l'\epsilon'/c &= MV_x + mv_x + \tau_1 h/d_1 \\ m\epsilon/c - m'\epsilon'/c &= MV_y + mv_y + \tau_2 h/d_2 \\ n\epsilon/c - n'\epsilon'/c &= MV_z + mv_z + \tau_3 h/d_3, \end{aligned} \quad (5)$$

where ϵ is the energy of the incident quantum of radiation, and ϵ' the total energy of the secondary radiation, M and m the masses, and V and v the velocities of the atom and electron, respectively. The energy, ϵ' , equals the $h\nu$ value of a critical absorption frequency characteristic of the chemical element in the crystal. In the secondary radiation from the atom it is divided into parts, $\epsilon_1, \epsilon_2, \dots$, corresponding to several emission lines in the characteristic X-ray spectrum ($\epsilon' = \epsilon_1 + \epsilon_2 + \dots$). The terms $\epsilon'l'/c$, $\epsilon'm'/c$ and $\epsilon'n'/c$ represent vector sums of the corresponding momenta.

According to our assumption the incident radiation transfers part of its momentum to the atom and electron without change of direction and we have, therefore, the equations:

$$\begin{aligned} (\epsilon/c - \epsilon'/c)l &= MV_x + mv_x, & (\epsilon/c - \epsilon'/c)m &= MV_y + mv_y, \\ & & (\epsilon/c - \epsilon'/c)n &= MV_z + mv_z. \end{aligned} \quad (6)$$

Subtracting these equations from equations (5) we get

$$\begin{aligned} \Sigma \epsilon_1/c(l - l_1) &= \tau_1 h/d_1, & \Sigma \epsilon_1/c(m - m_1) &= \tau_2 h/d_2, \\ & & \Sigma \epsilon_1/c(n - n_1) &= \tau_3 h/d_3. \end{aligned} \quad (7)$$

In these equations the first members consist of several terms. In each case the first term is very much larger than the sum of all the others. For example, in the K series of X-rays, the sum of the ϵ 's for all the lines in the other series amounts to less than 20% of the value of ϵ for the $K\alpha$ line. In the case of the $K\beta$ line, the sum of the ϵ 's other than the $K\beta$ amounts to only about 3.4% of the $K\beta$. For the γ line the sum amounts to a small fraction of 1% of the $K\gamma$. We see, therefore, that we may write the approximate equations

$$\epsilon_1/c(l - l_1) = \tau_1 h/d_1, \quad \epsilon_1/c(m - m_1) = \tau_2 h/d_2, \quad \epsilon_1/c(n - n_1) = \tau_3 h/d_3 \quad (8)$$

for the lines in the K series.

Since terms in equations (7) may be positive or negative, and since the value of a direction cosine can never exceed unity, it follows that l_1 , m_1 and n_1 , as given by equations (7), lie in the neighborhood and on both sides of certain mean values, given by equations (8). The approximations to the mean values are much closer in the case of the β lines than in that of the α lines, and still closer in the case of the γ lines. The equations (7), therefore, may be interpreted as meaning that the characteristic radiations come off from the crystal in directions approximating to those directions which obey the laws of defraction by the crystal. Further, the $K\beta$ reflection should be more intense and more sharply marked than the $K\alpha$ reflections, and the $K\gamma$ reflections should be still more sharply

marked. Again, in spectra of higher orders the direction cosines have larger values than in spectra of lower orders, and on this account we would expect somewhat more sharply marked reflections in the former than in the latter. It has been shown experimentally by Dr. Clark and the writer, that the characteristic reflections we discovered have precisely these peculiarities; so that our theory accounts for not only the reflections of rays from the crystal that are characteristic of the chemical elements in it but also for their most apparent details.

9. The application of the theory sketched in this note to the reflection, refraction, defraction, etc., of light, leads to interesting conclusions, some of which will now be discussed. The chief problem involved in these applications is the selection of suitable parameters, each having the dimensions of a length, to divide into the action constant, h .

Let us apply the theory to the case of a defraction grating. Let a quantum of radiation of energy $h\nu$ fall upon the grating in a direction making an angle i with the normal, and suppose that it is defracted at an angle r with the normal. Then, taking the components of the momenta in a direction along the surface of the grating perpendicular to its lines, we get the equation

$$(\sin i - \sin r)h\nu/c = nh/d,$$

where the distance, d , between the lines of the grating is the only distance in the direction we are considering that has any meaning. This equation reduces immediately to the ordinary equation for a grating:

$$n\lambda = d(\sin i - \sin r).$$

As a second problem, take the radiation reflected from a thin plate of thickness t . As before, let the radiation quantum fall on the surface at the angle i with the normal and suppose that inside of the plate its direction of motion makes the angle r with the normal. Let the velocity inside of the plate be c' , that outside c , and let the length of the plate be l . Taking components of the momentum parallel to the surface of the plate we have the equation:

$$h\nu/c \sin i - h\nu/c' \sin r = nh/l,$$

where l is the only distance in the direction considered that has any meaning. Unless l is very small and comparable with the wave-length of the incident ray, the second term of this equation practically equals 0 for all reasonable values of n , and the equation reduces to

$$\sin i : \sin r = c : c'$$

which is the normal equation for the refraction of light at the surface of two media.

Inside of the plate it may be that the radiation is deflected. Suppose that it is deflected and travels at an angle r' with the normal. Then, taking components parallel to the surface of the plates and in the direction through the plate, we have the equations

$$hv/c'(\sin r - \sin r') = n_1 h/l, \quad hv/c'(\cos r - \cos r') = n_2 h/t,$$

where t is the only length in a direction normal to the surface of the plate that has any meaning. From the first of these equations we deduce that $r' = r$ or $r' = 180^\circ - r$. In the first case, the quantum passes on through the plate without change of direction inside of it and in the second, it is deflected and comes back through the first surface of the plate at an angle equal to that at which it entered. In other words, it is reflected from the plate at an angle of reflection equal to the angle of incidence. Putting $r' = 180^\circ - r$, we find from the second equation that, $n_2 \lambda' = 2t \cos r$, λ' being the wave-length inside of the plate. This is the ordinary equation for the reflection of no light from a thin plate, producing so-called interference. The ordinary equation for the reflection of light may be obtained by adding a half quantum number.

10. In applying the principle under discussion to other problems of interference, the interpretation of the parameter to be used in the equations sometimes becomes very difficult. In general, interference equations are of the form $n\lambda$ equals the difference between two distances, and this may be transformed into an equation representing the difference between two quantities of radiation momentum.

11. According to our present theory the photoelectric effect should obey the law given by the energy equation

$$\epsilon - \epsilon' = \frac{1}{2}MV^2 + \frac{1}{2}mv^2, \quad (9)$$

in which V is given by equations (6) together with the direction cosine relation. Equation (9) takes account of the energy delivered to the atom as well as that delivered to the electron. The difference between it and Einstein's expression for the photoelectric effect does not amount to much, however, as this difference depends upon the ratio of m to M .

A similar correction should be applied to the equation giving the short wave-length limit of the continuous X-ray spectrum.