ator of order p in I. The conjugates of s^i under H are also the conjugates of this operator under G, and these conjugates generate an invariant subgroup of G whose order cannot exceed that of H. If p^l is the order of I the invariant subgroup H' generated by the p^{1-l} powers of all the operators of G has an order which cannot exceed that of H.

The number of the invariant subgroups of G which involve H' and have an order equal to that of H is of the form $1 + kp$ according to the theorem noted in the first paragraph. All of these subgroups give rise to non-cyclic quotient groups. The number of the invariant subgroups whose order is equal to that of H and which give rise to a cyclic quotient group must therefore be divisible by p, and this completes a proof of the italicized theorem of the preceding paragraph.

To prove the theorem noted in the first paragraph of this note it is now only necessary to count in the usual way each of the subgroups of order $p^{\alpha+1}$ contained in G as many times as it contains subgroups[3] of order p^α and to note that the sum thus obtained is equal to the sum obtained by counting each subgroup of order p^α as many times as it appears in a subgroup of order $p^{\alpha+1}$. When the quotient groups to which such a subgroup of the order p^α gives rise under the largest subgroup of G which transforms it into itself is non-cyclic the number of the subgroups of order $p^{\alpha+1}$ in which it appears is $1 + p + kp^2$. When this quotient group is cyclic there is a multiple of p such subgroups of order p^α and hence the total number is the same, modulo p^2, as if the number had been of the form $1 + p + kp^2$ in each case. Hence it follows by induction that the *number of the subgroups of order p^α contained in an arbitrary non-cyclic group of order p^m is of the form $1 + p + kp^2$, where α has an arbitrary value from 1 to $m-1$.*

[1] G. A. Miller, *Proc. London Math. Soc.*, **2**, (1905), p. 143.

[2] Miller, Blichfeldt, Dickson, *Finite Groups*, 1916, p. 128.

[3] *Ibid.*, p. 125.

THE INTERFERENCE OF LIGHT AND THE QUANTUM THEORY

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Communicated, May 29 ,1923

1. It is generally supposed that phenomena of interference are a grave objection to the theory of light quanta. They are usually considered as a support of the classical wave theory.

Prof. W. Duane[2] pointed out that if radiation momenta can be considered as transferred in quanta typical interference phenomena may be explained. He also showed that the converse proposition holds, viz., the experimentally known positions of maxima of light intensity and the hypothesis of light quanta lead to the conclusion that the most probable transfers of radiation momenta are quantized transfers.

In some cases the most probable transfers are the only possible ones. Such are the cases of a diffraction grating or of X-ray reflection from a crystal. In other instances as e.g., in Young's interference experiment the quantization locates only maxima of intensity. Thus a casual observer would be tempted to regard these latter cases as disproving the validity of Prof. Duane's principle of transfer of radiation momenta in quanta.

In what follows the writer believes to have shown that even some of the most classical of interference phenomena support the Principle of Quantized Radiation Momentum if a procedure analogous to an application of the Principle of Correspondence is made use of.

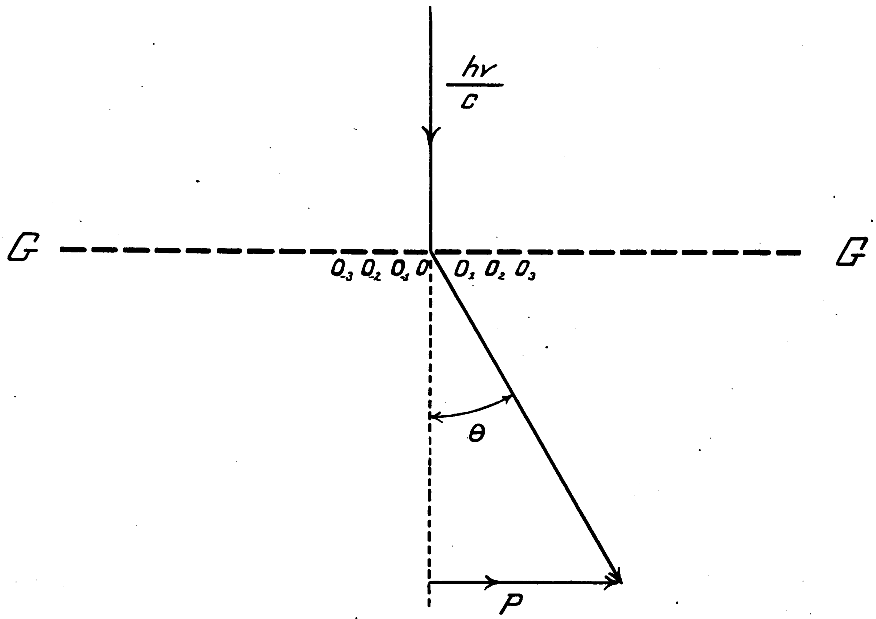

FIGURE 1

The theory is most easily applied to instruments of high resolving power. Thus the grating constitutes the simplest element from the point of view of quantum theory and other diffracting systems will be built out of gratings in our theory just as the classical wave theory builds everything out of point sources.

2. *The Diffraction Grating of Infinite Width.*—Consider the diffraction grating GG (Fig. 1) on which light quanta are incident normally. Some of the light quanta fall on the material between the holes of the grating

and are reflected or absorbed. Others penetrate through the holes. They either pass undeflected or a momentum p is imparted to them in a direction along the grating leaving the total energy of the quantum $h\nu$ practically unchanged. The momentum p we imagine to be due to the motion of something along the grating [say an electromagnetic disturbance or pulse]. If p is constant during this characteristic motion the values of p which are permitted by the quantum theory obey the usual quantum rule

$$ap = \int_0^a p dx = Nh \tag{1}$$

where a is the grating space. The right triangle with O as vertex [Fig. 1] gives now at once the angles through which quanta may be deflected. By (1)

$$\sin \theta = pc/h\nu = n\lambda/a \tag{2}$$

the usual form of the grating law. In general if the quantum is incident on the grating at an angle i we find that

$$\sin \theta + \sin i = n\lambda/a \tag{3}$$

agreeing again with the classical result.

In postulating the truth of (1) we suppose the grating to be infinitely wide for otherwise the characteristic motion along the grating is not strictly periodic.

The experimental facts about the intensities of the spectra of various orders when combined with a procedure analogous to an application of the Principle of Correspondence enables one to draw conclusions about the nature of the characteristic motion. Thus if spectra of different orders turn out to be of equal intensities [narrow grating lines] the Fourier expansion of the motion must contain to within the same range of values of N terms of equal magnitude.

3. *A Finite Number of Narrow, Parallel, Coplanar, and Equal Slits.*—Let us consider a system of m equal, coplanar, parallel, and narrow slits. We imagine the characteristic motion taking place across the slits. If we knew the exact manner in which each slit affects the characteristic motion we could find the resultant motion. This could then be expanded into a Fourier Integral. The meaning of a Fourier Integral is the same as that of a Fourier Series the period of which has been made infinitely great. If the period of the Fourier Series is very great it has always a definite period and consequently, if we should represent the Fourier Integral as the limit of a Fourier Series, we should be dealing with a motion which has a very long but finite period. We postulate that it is legitimate to apply physical considerations to this motion and to pass to the limit while making the period infinite.

In the Fourier Series representation the peculiarities caused in the motion by each of the slits are reproduced periodically at distances equal to the period of the expansion. The various reproductions of the system of slits we shall call images. Let the distance between homologous points of two successive images be a. Homologous slits of all the images form then a grating of grating space a. The characteristic motion corresponding to this set of slits is then of the type $\Sigma_n(b_n \cos 2\pi nct/a + a_n \sin 2\pi nct/a)$ where $a^2_m + b^2_m$ is a measure of the intensity of the mth order diffraction and may be imagined to be decreasing regularly with an increase in m. The nature of the dependence of $a^2_m + b^2_m$ on the angle of diffraction associated with it we shall suppose known. By so doing we do not restrict ourselves more than in the classical optical theory when an obliquity factor is introduced. The constant c is assumed to be the velocity of light.

Let the slits of the same image have distances $x_1, x_2, \ldots, x_m$ from an arbitrary point [+ when to the right]. We suppose that the characteristic motions due to either of these may be superposed in getting the motion as affected by all. Then the resultant motion is

$$y = \sum_{n=0}^{\infty} \sum_{i=1}^{m} \left[b_n \cos \frac{2\pi n(ct-x_i)}{a} + a_n \sin \frac{2\pi n(ct-x_i)}{a} \right] \quad (4)$$

The sum of the squares of the coefficients of the *cos* and *sin* terms in $2\pi nct/a$ is again proportional to the probability of the nth quantum transition, i. e., of a transition in which $n/a = (\sin \theta)/\lambda$. Substituting this value of n/a into (4) and considering the nth term only we see that the square of its amplitude is $(b^2_n + a^2_n) \left[\left(\sum_{i=0}^{m} \cos \tau_i \right)^2 + \left(\sum_{i=0}^{m} \sin \tau_i \right)^2 \right]$ where $\tau_i = 2\pi x_i \sin \theta/\lambda$. Remembering now that $b^2_n + a^2_n$ is the intensity which would be caused by a single slit we see that the above expression has precisely the classical value for the (amplitude)^2 of a vibration compounded of m vibrations having phase differences τ_i.

4. *A Slit of Finite Width.*—In the above treatment we have considered each slit as marking a period in the motion of the pulse that passes it. It becomes of interest to see whether it is possible to give a picture of the characteristic motion which will give values of a_n and b_n varying regularly and approximately equal for small values of n.

A simple type of motion which can be associated with a slit is one represented on figure 2 where A_1D_1 is the slit and O_1ABCDO_2 is the path. The Fourier Analysis of this case is represented essentially by

$$\frac{A\alpha}{\pi} \left[\frac{1}{2} + \sum_{m=1}^{\infty} \frac{\sin \frac{m\alpha}{2}}{\left(\frac{m\alpha}{2}\right)} \cos mx \right].$$

where $\alpha/2\pi$ is the ratio of $\overline{BC}$ to A, the latter denoting the distance between homologous points of successive images and where $\overline{AB} = A$. It is clear that for small values of m the amplitudes of the various harmonics become equal. To this extent the picture of figure 2 is satisfactory.

It appears, however, that if the theory here discussed is correct figure 2 represents the actual behavior of a slit. For if b should stand for the width of the slit, the probability of a quantum transition associated with the angle θ is proportional to

$$\left(\frac{\sin\frac{n\alpha}{2}}{\frac{n\alpha}{2}}\right)^2 = \left(\frac{\sin\frac{\pi b \sin\theta}{\lambda}}{\frac{\pi b \sin\theta}{\lambda}}\right)^2$$

where the substitutions $\alpha = 2\pi b/a$ and $n = a \sin\theta/\lambda$ have been made.

The path O_1ABCDO_2 appears to be the only type of path in which two neighboring slits when moved together so that their dividing line almost disappears give in the limit a path identical with the path of the two slits when their dividing line has completely disappeared.

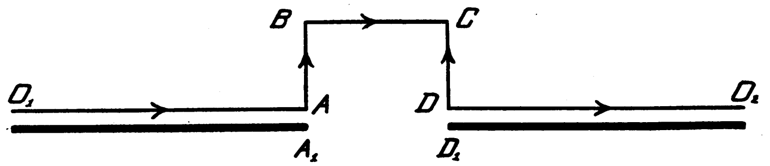

FIGURE 2

It is now clear that we are able to treat any system of coplanar parallel slits.

5. *Relation to Classical Diffraction Theory.*—In the above we made use of three assumptions: (a) the quantum law of a wide grating; (b) the possibility of representing a single slit or a narrow grating as one element of a wide grating of a large grating space; (c) a mathematical postulate as to probabilities of transitions which is analogous to Bohr's Principle of Correspondence. Assumption (a) is the fundamental one and may be discussed without the aid of (b) in the case of a wide grating.

There is an essential difference between our points of view on diffraction in the classical and in the quantum theory. In the classical wave theory we consider a wave train falling on the grating and by using the wave theory of light we find that the energy of the incident train reappears after diffraction in certain portions of space which at some distance away from the grating form a fan like structure of planes. Thus the incident energy is thought of as subdivided into parts. It must be conceded that this classical conception when applied to light is very conventional inasmuch as the existence of energy in space can concern us only if we detect this energy.

In the quantum theory the subdivision of $h\nu$ into smaller parts can also be imagined while it is in empty space. However such a subdivision will not be detected experimentally and will have the character of an abstract theory. For this reason it is more convenient to think only of the facts which have a bearing on our experiments and to operate with Einstein's light quanta.

Nevertheless it is useful to avail ourselves of the wave theory in discussing the laws of light quanta. We can try, for example, to construct a mechanism with some classical features which will involve some of the elements of quantum theory. Let us imagine a hydrogen atom H_1 at the focus of a parabolic reflector P_1 and a hydrogen atom H_2 at the focus a parabolic reflector P_2. The grating GG is placed so as to diffract rays of frequency ν directed along the axis of P_1 into rays along the axis of P_2. The axis of P_1 is perpendicular to GG and that of P_2 is oblique. H_1 is in the excited state. Its electron falls from an outer orbit into the innermost. A quantum $h\nu$ is emitted. This is absorbed by H_2 and brings H_2 into the excited state. During the process of absorption by H_2 a wave is emitted by H_2 and this travels back to GG where it is also diffracted. The wave emitted by H_1 has a momentum normal to GG. The momentum of the radiation from H_2 has a component parallel to GG. This component takes the place of the quantized radiation momentum.

We now can find an interpretation of our application of the Principle of Correspondence. The wave from H_2 on reaching GG produces secondary wavelets and these travel along GG with the velocity of light. They are connected with electrical disturbances in GG—say motions of electrons. As the whole pulse moves across the grating its head takes in new grating lines. Thus the line AB of figure 2 could be thought of as the acceleration of the electron at the head of the pulse.

These accelerations may be considered as having equal weights for all of the lines of the grating only if distant points are considered. Thus the theory may not be logically applied to Fresnel diffraction phenomena. Perhaps it will be possible to treat these by means of the adiabatic hypothesis or a more detailed application of the correspondence principle.

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[2] These PROCEEDINGS, May, 1923.