Traces, ideals, and arithmetic means

Victor Kaftal[†] and Gary Weiss

Department of Mathematical Sciences, University of Cincinnati, Cincinnati, OH 45221-0025

Edited by Richard V. Kadison, University of Pennsylvania, Philadelphia, PA, and approved March 29, 2002 (received for review February 7, 2002)

This article grew out of recent work of Dykema, Figiel, Weiss, and Wodzicki (Commutator structure of operator ideals) which *inter alia* characterizes commutator ideals in terms of arithmetic means. In this paper we study ideals that are arithmetically mean (am) stable, am-closed, am-open, soft-edged and soft-complemented. We show that many of the ideals in the literature possess such properties. We apply these notions to prove that for all the ideals considered, the linear codimension of their commutator space (the "number of traces on the ideal") is either 0, 1, or ∞. We identify the largest ideal which supports a unique nonsingular trace as the intersection of certain Lorentz ideals. An application to elementary operators is given. We study properties of arithmetic mean operations on ideals, e.g., we prove that the am-closure of a sum of ideals is the sum of their am-closures. We obtain cancellation properties for arithmetic means: for principal ideals, a necessary and sufficient condition for first order cancellations is the regularity of the generator; for second order cancellations, sufficient conditions are that the generator satisfies the exponential Δ_2-condition or is regular. We construct an example where second order cancellation fails, thus settling an open question. We also consider cancellation properties for inclusions. And we find and use lattice properties of ideals associated with the existence of "gaps."

The algebra B(H) of bounded linear operators on a separable, infinite-dimensional, complex Hilbert space has only one nonzero proper closed two-sided ideal, the class of compact operators $K(H)$. There is, however, a rich structure of nonclosed two-sided ideals of B(H) (operator ideals). Their study was initiated by Calkin (1), who established a lattice isomorphism between ideals and characteristic sets, i.e., the hereditary (solid) positive cones $\Sigma \subset c_o^*$ (the collection of monotone sequences decreasing to 0) that are invariant under ampliation: $\xi \to (\xi_1, \xi_1, \xi_2, \xi_2, \xi_3, \xi_3, \ldots)$. Given an ideal I, call $\Sigma(I) := \{s(X) | X \in I\}$ the characteristic set of I where $s(X) := \langle s_n(X)\rangle$ is the sequence of s-numbers of X, i.e., the eigenvalues of $|X|$ counting multiplicities and arranged in decreasing order with infinitely many zeroes added in case X is finite rank. Conversely, if Σ is a characteristic set, the diagonal operators diag ξ with $\xi \in \Sigma$ generate the (unique) ideal I such that $\Sigma(I) = \Sigma$.

For each ideal I, we denote by $[I, B(H)]$ the commutator space for I (also known as the commutator ideal), i.e., the linear span of all the commutators XB-BX where $X \in I$ and $B \in B(H)$. Commutator spaces are central to the theory of operator ideals. For instance, they play a key role in defining traces (see the next section). Starting with Halmos (2) and Pearcy and Topping (3), a great deal of effort has been devoted over the years to characterizing commutator spaces for various ideals (see ref. 4 for a comprehensive list of references). The line of research leading to this paper began with a result of one of the authors (5): for the trace class ideal Λ_1 with the usual trace Tr, the commutator space $[\mathfrak{L}_1, B(H)]$ is strictly contained in ker Tr $= \{X \in \mathfrak{L}_1 | \text{Tr } X = 0\}$. The key test case was the diagonal operator $X = \text{diag}(-1, d_1, d_2, \ldots)$ where $d_n \downarrow 0$ and $\Sigma_{n=1}^{\infty} d_n = 1$. Then $X \in [\mathfrak{L}_1, B(H)]$ if and only if $\Sigma_{n=1}^{\infty} d_n \log n < \infty$. Notice that this condition is equivalent to asking that the arithmetic mean of the sequence $\lambda := (-1, d_1, d_2, \ldots)$ be itself summable. Kalton (6) characterized $[\mathfrak{L}_1, B(H)]$ this way in terms of arithmetic means of eigenvalue sequences. For an arbitrary sequence $\lambda = \langle\lambda_n\rangle$, denote by λ_a its arithmetic mean sequence, namely, $\lambda_a := \langle(1/n)\ \Sigma_{j=1}^{n}\ \lambda_j\rangle$.

Dykema, Figiel, Weiss and Wodzicki proved the following (see ref. 4, Theorem 5.6, and Introduction formula 2).

Theorem 1. For any proper ideal I, if $X \in I$ is a normal operator with λ its sequence of eigenvalues counting multiplicities and ordered according to decreasing moduli, then $X \in [I, B(H)]$ if and only if $\lambda_a \in \Sigma(I)$.

This result, along with others in ref. 4, have consequences in the area of operator ideals and traces. Here we explore some of these consequences focusing on a number of questions.

How many traces can an ideal support? We found that for all the ideals in the literature that we considered, the answer is either 0, 1, or ∞; 0 can occur only when diag $\omega \in I$ ($\omega := \langle 1/n\rangle$ will denote the harmonic sequence), and 1 can occur if diag $\omega \notin I$. In the latter case, we determined the largest ideal possessing a unique trace. Our analysis here rests partly on the notions of soft-edged and soft-complemented ideals (see *Definition 6*), which include many of the ideals in the literature.

What are the implications for operator theory? Applications are given to elementary operators: Fuglede–Putnam type results and a question of Shulman.

As seen in Theorem 1 and throughout refs. 4 and 6, the arithmetic mean operation plays a critical role in the theory of commutators and of operator ideals. To make this role more transparent, given an ideal I, the associated arithmetic mean ideal I_a and, respectively, the pre-arithmetic mean ideal ${}_aI$ are defined in ref. 4 as:

$$\Sigma(I_a) := \{\xi \in c_o^* | \xi \le \eta_a \text{ for some } \eta \in \Sigma(I)\}$$

$$\Sigma({}_aI) := \{\xi \in c_o^* | \xi_a \in \Sigma(I)\}.$$

So, for instance, a special case of Theorem 1 is that $[I, B(H)]^+ = ({}_aI)^+$ where $({}_aI)^+$ denotes the class of positive operators in the ideal ${}_aI$. The *arithmetic mean* (*am*) *closure*, respectively, *am-interior*, of an ideal I are defined in ref. 4 as $I^- := {}_a(I_a)$ and $I^o := ({}_aI)_a$ and play an important role in the theory. Indeed, many ideals in the literature are am-closed, i.e., $I = I^-$. This motivates us to investigate questions on am-closure, am-closed ideals, and their properties.

One question is whether the sum of am-closed ideals is am-closed. We prove that it is by showing that: $(I + J)^- = I^- + J^-$. The proof combines weak majorization theory, convexity, and stochastic matrices (extended to infinite sequences and to notions of infinite convexity).

Another set of questions relate to cancellation properties of arithmetic means: for which ideals I does $J_a = I_a$, $J_a \subset I_a$, and $J_a \supset I_a$ imply, respectively, $J = I$, $J \subset I$, and $J \supset I$? And similarly for ${}_aI$? We answer these "first order cancellation" questions when I is principal. If X is a generator of I and $\pi = s(X)$, denote $I = (\pi)$. Then we prove (*Theorem 11*) that $J_a = (\pi)_a$ (or ${}_aJ = {}_a(\pi)$) implies $J = (\pi)$ if and only if π is *regular*, i.e., $\pi \asymp \pi_a$, or equivalently, $(\pi) = (\pi)_a$. Here the equivalence of two sequences $\xi \asymp \eta$, means that both $\xi = O(\eta)$ and $\eta = O(\xi)$. Theorem 11

This paper was submitted directly (Track II) to the PNAS office.

[†]To whom reprint requests should be addressed. E-mail: kaftal@math.uc.edu.

depends on properties of the lattice of principal ideals with regular generators. Harder, even for the principal ideal case, are second order cancellation questions: for which π does $J_{a^2} = (\pi)_{a^2}$, $J_{a^2} \subset (\pi)_{a^2}$, and $J_{a^2} \supset (\pi)_{a^2}$ imply, respectively, $J_a = (\pi)_a$, $J_a \subset (\pi)_a$, and $J_a \supset (\pi)_a$? The first and second implications but not the third are true for π regular. Denoting $h(\pi) := \pi_a/\pi$, it turns out that sufficient conditions can be obtained in terms of the sequence $h(\pi_a) = \pi_{a^2}/\pi_a$. If $h(\pi_a)$ is equivalent to a monotone sequence, then the second implication holds (*Proposition 13*). Moreover, if $\sigma(\pi) := \langle \Sigma_{j=1}^{n} \pi_j \rangle$ satisfies an exponential Δ_2-condition (see paragraph after *Proposition 13*) or equivalently, $h(\pi_a) \asymp \langle \log n \rangle$, then all three cancellations hold (*Proposition 14*). However, in general, second order cancellation can fail: we construct a pair $\xi, \pi \in c_o^*$ such that $(\xi)_{a^2} = (\pi)_{a^2}$ but $(\xi)_a \neq (\pi)_a$. This settles in the negative a question of M. Wodzicki.

Traces

The natural domain of the usual trace Tr on B(H) is the trace class $\mathfrak{L}_1$. However, ideals can support more exotic traces. Traces are unitarily invariant linear functionals on an ideal I. Equivalently, they are linear functionals vanishing on the commutator space $[I, B(H)]$, i.e., elements of the linear dual to the quotient $I/[I, B(H)]$. In general, they are not assumed to be positive or faithful. A trace that vanishes on the ideal F of finite rank operators (a subspace of all nonzero ideals) is called singular, and nonsingular otherwise. The first example of a (positive) singular trace was given by Dixmier (7). Its natural domain $\mathfrak{S}_\Omega$ (although Dixmier's construction was somewhat more general) is the dual of the Macaev ideal $\mathfrak{S}_\omega$ (8), and is defined as the ideal with characteristic set $\{\xi \in c_o^* | \langle \Sigma_{j=1}^{n} \xi_j \rangle = O(\log n)\}$. $\mathfrak{S}_\Omega$ and $\mathfrak{S}_\omega$ are denoted by Connes in ref. 9 by $\mathfrak{L}^{(1,\infty)}$ and $\mathfrak{L}^{(\infty,1)}$, respectively, and in ref. 4 by $M(1/\omega_a)$ and $\mathfrak{L}(\log)$, respectively. In the notation of ref. 4 and of this paper, $\mathfrak{S}_\Omega$ coincides with the am-closure $(\omega)^-$ of the principal ideal (ω) generated by diag ω, i.e., $\Sigma(\mathfrak{S}_\Omega) = \{\xi \in c_o^* | \xi_a = O(\omega_a)\}$.

The am-closure $(\pi)^- = {}_a((\pi)_a)$ of an arbitrary principal ideal (π) plays an important role in the theory of operator ideals. Denoted $\mathfrak{S}_\Pi$ by Gohberg and Krein (10), $(\pi)^-$ was shown to support the complete norm $\|X\| := \sup\, (s(X)_a)_n/(\pi_a)_n$. Gohberg and Krein noticed that when π is regular, i.e., $\pi \asymp \pi_a$, then $(\pi)^- = (\pi)$. Varga (11) proved that an ideal (π) supports a nontrivial positive trace precisely when $(\pi)^-$ does, and this holds if and only if $(\pi) \neq (\pi)^-$ or equivalently, when π is irregular. Clearly, if π is regular then so is π_a. As a consequence of Varga's work, it turns out that if π_a is regular, so is π. Another proof of this fact is given in ref. 4 (Theorem 3.10). We found a quantitative version of the same result: for every n, there is an $m > n$ (actually we can specify that $m \leq nh(\pi)_n$ where $h(\pi) = \pi_a/\pi$) such that $\frac{1}{2} \log h(\pi)_n \leq h(\pi_a)_m$, and this inequality is sharp.

For general ideals, by Theorem 1, $[I, B(H)]^+ = ({}_aI)^+$ and thus ${}_aI$ is the largest ideal contained in $[I, B(H)]$. Since I is also the smallest ideal containing $[I, B(H)]$, ${}_aI = [I, B(H)]$ if and only if ${}_aI = I$. From the chain of inclusions:

$$ {}_aI \subset I^o = ({}_aI)_a \subset I \subset {}_a(I_a) = I^- \subset I_a, $$

we see that ${}_aI = I$ if and only if $I_a = I$. An ideal I for which $I = {}_aI = I_a$ is called *arithmetically mean stable* (*stable* for short). Thus stable ideals are precisely those with no nonzero traces (see ref. 4).

An important consequence of Theorem 1 is that an ideal I supports a nonsingular trace if and only if diag $\omega \notin I$, which condition is equivalent to $[I, B(H)]^+ = \{0\}$ and which in turn is equivalent to $[I, B(H)] \cap F \subset \{X \in F | \mathrm{Tr}\, X = 0\}$. This permits one to extend uniquely Tr from F to $F + [I, B(H)]$ and then to I by a Hamel basis argument (nonuniquely when the containment $F + [I, B(H)] \subset I$ is proper). Similarly, if I contains the trace class, Tr can be extended to I if and only if diag $\omega \notin I$. As a further consequence of Theorem 1 we obtain Proposition 2.

Proposition 2. If diag $\omega \notin I$, then $(\mathfrak{L}_1 + [I, B(H)])^+ = \mathfrak{L}_1^+$.

Notice that the converse is not true since $[(\omega), B(H)]^+ = \mathfrak{L}_1^+$.

Applications to Elementary Operators

Trace extensions find natural applications to questions on elementary operators. If $A_i, B_i \in B(H)$, then the map $B(H) \ni T \to \Delta(T) := \Sigma_{i=1}^{n} A_iTB_i$ is called an elementary operator and the adjoint map is $\Delta^*(T) := \Sigma_{i=1}^{n} A_i^*TB_i^*$. Elementary operators include commutators and intertwiners and hence their theory is connected to commutator spaces. The Fuglede–Putnam Theorem (12, 13) states that for the case $\Delta(T) = AT\text{-}TB$ with A, B normal, $\Delta(T) = 0$ implies $\Delta^*(T) = 0$. For $n = 2$, Weiss (14) generalized this to the case where $\{A_i\}$ and $\{B_i\}$ are separately commuting families of normal operators by proving that $\Delta(T) \in \mathfrak{L}_2$ implies $\Delta^*(T) \in \mathfrak{L}_2$ and that $\|\Delta(T)\|_{\mathfrak{L}_2} = \|\Delta^*(T)\|_{\mathfrak{L}_2}$. [This is also a consequence of Voiculescu's (15) Theorem 4.2 and Introduction to Section 4.] Shulman (16) proved that for $n = 6$, $\Delta(T) = 0$ does not imply $\Delta^*(T) \in \mathfrak{L}_2$.

If we impose some additional ideal-type conditions on the elementary operator Δ and/or on T, we can extend these implications to arbitrary n past the obstruction found by Shulman and the limitations of ref. 15. Assume there is an ideal I not containing diag ω but containing $\mathfrak{L}_1$ such that the product $(A_iT)(B_i)$ of the principal ideals (A_iT) and (B_i) is contained in $I^{1/2}$ for all i (resp., $(A_i)(TB_i) \subset I^{1/2}$ for all i). This includes, for instance, the cases where for each i, at least one of the operators A_i, B_i, T is in $I^{1/2}$; or at least two are in $I^{1/4}$; or all three are in $I^{1/6}$. The usefulness of these conditions lies in the fact that the ideal I can be "much larger" than $\mathfrak{L}_1$. Then by using the general identity $[I, J] = [IJ, B(H)]$ from ref. 4 (Theorem 5.10), we obtain that $|\Delta^*(T)|^2 - |\Delta(T)|^2 \in [I, B(H)]$ (resp., $|(\Delta^*(T))^*|^2 - |(\Delta(T))^*|^2 \in [I, B(H)]$). Thus, if $\Delta(T) \in \mathfrak{L}_2$, it follows that $|\Delta^*(T)|^2$ is in $(\mathfrak{L}_1 + [I, B(H)])^+$ and, by Proposition 2, that $\Delta^*(T) \in \mathfrak{L}_2$. Moreover, since diag $\omega \notin I$, Tr can be extended to I and thus its extension must vanish on $[I, B(H)]$. Therefore $\|\Delta(T)\|_{\mathfrak{L}_2} = \|\Delta^*(T)\|_{\mathfrak{L}_2}$ and, in particular, $\Delta(T) = 0$ implies $\Delta^*(T) = 0$.

A further application is to a problem considered by Shulman. Here the operators $\{A_i\}$ and $\{B_i\}$ are not necessarily commuting nor normal. He showed that $\Delta^*\Delta(T) = 0$ does not imply $\Delta(T) = 0$ and conjectured that this implication holds under the additional assumption that $\Delta(T) \in \mathfrak{L}_2$. Reasoning as in the previous case, if there is an ideal I not containing diag ω such that $(A_iT)(B_i) \subset I^{1/2}$ for all i (resp., $(A_i)(TB_i) \subset I^{1/2}$ for all i), then $|\Delta(T)|^2 - T^*\Delta^*\Delta(T) \in [I, B(H)]$ (resp., $|(\Delta(T))^*|^2 - T(\Delta^*\Delta(T))^* \in [I, B(H)]$). Hence, if $\Delta^*\Delta(T) \in \mathfrak{L}_1$, we obtain that $\Delta(T) \in \mathfrak{L}_2$ and that $\|\Delta(T)\|_{\mathfrak{L}_2}^2 = \mathrm{Tr}\, T^*\Delta^*\Delta(T)$. In particular, if $\Delta^*\Delta(T) = 0$ it follows that $\Delta(T) = 0$.

Uniqueness of Traces

For all $0 \neq X \in \mathfrak{L}_1$, $s(X)_a \asymp \omega$. So, for $X \in \mathfrak{L}_1^+$, instead of the arithmetic mean, the relevant operation is the *arithmetic mean at infinity* $X_{a_\infty} := \mathrm{diag}\, \langle (1/n) \Sigma_{n+1}^{\infty} s_j(X) \rangle$ (see ref. 4, formula 17 and Theorem 5.11). A special case of Theorem 5.11 (iii) is that if $I \subset \mathfrak{L}_1$, then $[I, B(H)]$ contains all the operators in I with zero trace if and only if I is invariant under the arithmetic mean at infinity. Further using Theorem 1, we obtain Proposition 3.

Proposition 3. If I is an ideal not containing diag ω, then $(F + [I, B(H)])^+ = \{X \in \mathfrak{L}_1^+ | X_{a_\infty} \in I\}$.

As a consequence, $(F + [I, B(H)])^+$ is always hereditary (solid). Notice that an ideal I not containing diag ω has a unique trace (up to scalar multiplication), i.e., $\dim I/[I, B(H)] = 1$, if and only if $I = F + [I, B(H)]$. It is easy to verify that $X \in$

MATHEMATICS

$(F + [\mathfrak{L}_1, \mathrm{B(H)}])^+$, i.e., $X_{a_\infty} \in \mathfrak{L}_1$, if and only if $X \in \mathfrak{L}(\sigma(\log))^+$. Here $\mathfrak{L}(\sigma(\log))$ is the Lorentz ideal with characteristic set $\{\xi \in c_o^* | \Sigma\, \xi_n \log n < \infty\}$ and $\sigma(\lambda) := \langle \Sigma_{j=1}^n \lambda_j \rangle$ denotes the initial partial sum sequence of λ (see ref. 4, 2.25). Similarly, one obtains $X_{a_\infty} \in \mathfrak{L}(\sigma(\log^p))$ (the Lorentz ideal with characteristic set $\{\xi \in c_o^* | \Sigma\, \xi_n \log^p n < \infty\}$ if and only if $X \in \mathfrak{L}(\sigma(\log^{p+1}))^+$. Notice that from Proposition 3 it follows that $\mathfrak{L}(\sigma(\log))^+ = (F + [\mathfrak{L}_1, \mathrm{B(H)}])^+$. Another useful consequence of Proposition 3 is that if diag $\omega \notin I$, then I has a unique trace, i.e., $I = F + [I, \mathrm{B(H)}]$, if and only if $X_{a_\infty} \in I$ for every $X \in I^+$. Consequently all such I are contained in $\bigcap_{p=0}^{\infty} \mathfrak{L}(\sigma(\log^p))$. Thus, we obtain Theorem 4.

Theorem 4. $\bigcap_{p=0}^{\infty} \mathfrak{L}(\sigma(\log^p))$ is the largest ideal (not containing diag ω) with a unique trace, and that trace is Tr.

In particular, $\mathfrak{L}_1$ has no unique trace, not even a unique nonsingular trace. If I is the principal ideal generated by an operator $X \in \mathfrak{L}_1$, the conditions of Theorem 1 are satisfied if and only if for some $c > 0$, $X_{a_\infty} \le c$ diag s(X). In ref. 18, Corollary 7, Kalton proves that an operator $X \in \mathfrak{L}_1$ is uniquely traceable, i.e., I supports a unique separately continuous trace (see ref. 18 for the definition) if and only if there is a $p > 1$ and $c > 0$ such that $s_m(X) \le c(m/n)^{-p}\, s_n(X)$. As this condition clearly implies $X_{a_\infty} \le c'$ diag s(X) for some $c' > 0$, we see that I supports only one separately continuous trace precisely when it supports only one trace, namely, the usual trace Tr.

Dimension of $I/[I, \mathrm{B(H)}]$

Beyond the question of uniqueness of traces, it is natural to ask "how many" traces are supported by an ideal I or, equivalently, what are the possible values of dim $I/[I, \mathrm{B(H)}]$?

As mentioned in ref. 4 (5.27-Remark 1), Dixmier's method in constructing nonsingular traces can be used to prove that $\dim(\pi)^-/[(\pi)^-, \mathrm{B(H)}] = \infty$ whenever $\pi = o(\pi_a)$. The condition $s(Y) = o(s(X))$ for $X, Y \in K(H)$ (equivalently, $Y = KX$ for some $K \in K(H)$) also plays an important role in Varga's treatment of traces on principal ideals and their am-closures. When $s(Y) = o(s(X))$, we can interpolate between Y and X any number of operators. This leads to:

Lemma 5. If there exist $X \in I$ and $K \in K(H)$ such that $0 \le KX \notin F + [I, \mathrm{B(H)}]$, then $I/(F + [I, \mathrm{B(H)}])$ has uncountable dimension.

Considering for which kind of ideals the existence of a positive $X \notin F + [I, \mathrm{B(H)}]$ guarantees that we can always find a $K \in K(H)$ for which we also have $0 \le KX \notin F + [I, \mathrm{B(H)}]$ leads us to the definitions:

Definition 6. We call an ideal I *soft-edged* if $I = K(H)I$ and *soft-complemented* if, for any ideal J, the condition $K(H)J \subset I$ implies $J \subset I$.

Thus, an ideal I is soft-edged if for every $\pi \in \Sigma(I)$ there is a $\xi \in \Sigma(I)$ such that $\pi = o(\xi)$. An ideal I is soft-complemented if for every $c_o^* \ni \pi \notin \Sigma(I)$ there is a $c_o^* \ni \xi \notin \Sigma(I)$ such that $\xi = o(\pi)$.

The main ideals in the literature are all either soft-edged or soft-complemented. We prove that countably generated ideals are soft-complemented. They are not necessarily soft-edged. Indeed, if π satisfies the Δ_2-*condition*, i.e., $\sup \pi_n/\pi_{2n} < \infty$, then the principal ideal (π) is not soft-edged. Lorentz ideals are both soft-edged and soft-complemented. If an ideal I is soft-complemented, its pre-arithmetic mean ideal ${}_aI$ is also soft-complemented. Thus all Marcinkiewicz ideals (the prearithmetic means of principal ideals) are soft-complemented. If M is a nondecreasing function, the associated Orlicz ideal $\mathfrak{L}_M$ (respectively, small Orlicz ideal $\mathfrak{L}_M^{(o)}$) are the ideals with characteristic set $\{\xi \in c_o^* |\ \Sigma_n M(t\xi_n) < \infty$ for some $t > 0\}$ (respectively, $\{\xi \in c_o^* | \Sigma_n M(t\xi_n) < \infty$ for all $t > 0\}$) (see ref. 4, 2.37 and 4.7). We show that small Orlicz ideals are soft-edged and Orlicz ideals are soft-complemented. The Gohberg and Krein ideals $\mathfrak{S}_\phi$ generated by symmetric norming functions ϕ are always soft-complemented and they are soft-edged if and only if ϕ is mononormalizing (see ref. 10 for the definitions).

Theorem 7. If I is soft-edged or soft-complemented then dim $I/(F + [I, \mathrm{B(H)}])$ is either 0 or ∞.

When this dimension is 0, dim $I/[I, \mathrm{B(H)}]$ is either 1 or 0 according to whether diag $\omega \notin I$ or diag $\omega \in I$. A further consequence of Lemma 5 is that if diag $\omega \notin I$ and $I \not\subset \bigcap_{p=0}^{\infty} \mathfrak{L}(\sigma(\log^p))$ (so, in particular, if $I \not\subset \mathfrak{L}_1$) then dim $I/(F + [I, \mathrm{B(H)}]) = \infty$. The key fact in this argument is that the ideals $\mathfrak{L}(\sigma(\log^p))$ are soft-complemented.

Of course not all ideals are either soft-edged or soft-complemented. For instance, if I is soft-complemented but not soft-edged, then any ideal J strictly between $K(H)I$ and I is neither soft-edged nor soft-complemented. The simplest example of such a situation is when $I = (\pi)$ is the principal ideal for some π satisfying the Δ_2-condition. Then $\Sigma(K(H)I) = o(\pi) \cap c_o^*$, $\Sigma(I) = O(\pi) \cap c_o^*$, and there are infinitely many ideals lying between $K(H)I$ and I. This example leads us to Definition 8.

Definition 8. Given an ideal I, let $se(I) = K(H)I$ and let $sc(I)$ be the ideal with characteristic set $\{\xi \in c_o^* | o(\xi) \cap c_o^* \subset \Sigma(I)\}$.

Then, $se(I)$ is the largest soft-edged ideal contained in I and $sc(I)$ is the smallest soft-complemented ideal containing I or, equivalently, it is the largest ideal J for which $K(H)J \subset I$.

It follows that $sc(se(I)) = sc(I)$ and $se(sc(I)) = se(I)$. As a consequence, if $se(I) \subset J \subset sc(I)$, then $se(J) = se(I)$ and $sc(J) = sc(I)$. This leads us to Theorem 9.

Theorem 9. If either $se(I)/[se(I), \mathrm{B(H)}]$ or $sc(I)/[sc(I), \mathrm{B(H)}]$ has infinite dimension, then both have infinite dimension. In this case, $I/[I, \mathrm{B(H)}]$ has infinite dimension.

In particular, $se(I)$ is stable, i.e., $se(I) = [se(I), \mathrm{B(H)}]$, if and only if $sc(I)$ is stable. Applying these ideas to Orlicz ideals, one first shows that $se(\mathfrak{L}_M) = \mathfrak{L}_M^{(o)}$ and that $sc(\mathfrak{L}_M^{(o)}) = \mathfrak{L}_M$. Then if $\mathfrak{L}_M \ne \mathfrak{L}_M^{(o)}$ and if either of the two ideals is not stable, then the other one is not stable (see ref. 4, Theorem 5.26), in which case, for every ideal I between $\mathfrak{L}_M^{(o)}$ and $\mathfrak{L}_M$, it follows that dim $I/[I, \mathrm{B(H)}] = \infty$.

Am-closure, Am-Interior, and Related Ideals

We have seen that the notion of am-closure plays a critical role for traces and norms on principal ideals. This notion was studied in ref. 4 for general ideals where it turned out to be relevant also in the analysis of single commutators (see ref. 4, Theorem 7.3 and Corollary 7.10).

The Gohberg and Krein ideals $\mathfrak{S}_\phi$ generated by symmetric norming functions ϕ are am-closed, and they include many ideals in the literature such as Lorentz ideals for concave functions, Marcinkiewz ideals, and Orlicz ideals for convex functions. (See ref. 4, 4.7 and 4.9.)

Majorization theory is useful in investigating am-closed ideals. By definition, an ideal is am-closed, if and only if for $\xi \in c_o^*$, $\xi_a \le \eta_a$ for some $\eta \in \Sigma(I)$ implies $\xi \in \Sigma(I)$. In other words, the characteristic set $\Sigma(I)$ of an am-closed ideal I is hereditary under weak majorization (the Hardy–Littlewood–Polya–Schur order). We prove that this is equivalent to the conditions that $\Sigma(I)$ contains all summable monotone sequences and is invariant under direct sums of block doubly stochastic finite matrices followed by monotonizing. In other words, if $\xi \in \Sigma(I)$ and $P = \Sigma \bigoplus P_k$ is a direct sum of doubly stochastic finite matrices, i.e., matrices with nonnegative entries with rows and columns summing to 1, then the monotonization $(P\xi)^*$ of $P\xi$ must be in $\Sigma(I)$. This implies but is not equivalent to the conditions that $\Sigma(I)$ contains all summable monotone sequences and that $\Sigma(I)$ is invariant under infinite convex combinations of infinite permutation matrices followed by monotonizing.

A useful application of this result is that: $(I + J)^- = I^- + J^-$ for arbitrary pairs of ideals. Since directed unions of am-closed ideals are am-closed, this identity shows that every ideal I contains a largest am-closed ideal which we denote by I_-. We obtain analogous results for am-open ideals, i.e., that every ideal I is contained in a smallest am-open ideal I^{oo} and that $(I + J)^{oo} = I^{oo} + J^{oo}$. Notice that $(I + J)^o \supset I^o + J^o$ and $(I + J)_- \supset I_- + J_-$ but both inclusions can be proper.

For a principal ideal $I = (\pi)$, both I^o and I^{oo} are principal and have generators diag π^o and diag π^{oo}, respectively. Here π^o is (up to equivalence) the largest average $\xi_a \leq \pi$, and π^{oo} is the smallest average $\xi_a \geq \pi$ (no equivalence is necessary here). To identify $(\pi)_-$, we prove that if $\xi \in c_o^*$ and $\zeta_a \leq \xi_a$ for $\zeta \in c_o^*$ implies $\zeta \leq \pi$, then $\xi_a \leq \pi$. In other words, $(\xi)^- \subset (\pi)$ implies $(\xi)_a \subset (\pi)$ and hence $(\xi)^- \subset {}_a(\pi)$. As ${}_a(\pi)$ is am-closed, this yields:

Theorem 10. If I is a principal ideal, then $I_- = {}_aI$.

Theorem 10 has several consequences. First of all, it yields a new proof of the fact that a principal ideal is am-closed if and only if it is stable (11) (see also ref. 4, Theorem 5.20). Notice that the am-closure of an ideal is principal if and only if the ideal is principal. This follows from the fact mentioned earlier that for any $\pi \in c_o^*$, π is regular if and only if π_a is regular.

Further consequences of Theorem 10 are obtained by exploiting lattice properties of ideals, in particular, of some classes of principal ideals. Blass and Weiss (19) proved that $K(H)$ is the sum of two proper ideals (neither of which can be countably generated) and in general, every ideal that properly contains F is the sum of two proper ideals. Here we obtain that with respect to the inclusion order, the lattice of principal ideals has no "gaps", that is, between any two principal ideals lies another one. Notice this is not true in general, e.g., below every stable principal ideal (π) there is a gap between (π) and a maximal ideal in (π) not containing diag π. Such a maximal ideal must necessarily also be stable but cannot be principal.

A principal ideal has a unique generator up to s-sequence equivalence if and only if any (and hence all) of its generators have their s-sequence satisfying the Δ_2-condition, in short, a Δ_2-generator. We obtain that between an ideal with a Δ_2-generator and another comparable principal ideal (whether contained in it or containing it) lies a principal ideal with a Δ_2-generator. The same holds replacing Δ_2-generators with regular generators: between two comparable principal ideals, one of which has a regular generator, i.e., is stable, lies another principal stable ideal.

Cancellation Properties for Arithmetic Means-First Order

In studying the arithmetic mean operations on ideals it is natural to consider cancellation properties of the kind: for which ideals I does $I_a = J_a$ imply $I = J$? And similarly, when does ${}_aI = {}_aJ$ imply $I = J$? Notice first that $I_a = (I^-)_a$. So for the first question, a necessary condition is that I be am-closed (though not sufficient since $\mathfrak{L}_1$ is am-closed and $(\mathfrak{L}_1)_a = (F)_a$). As the examples following Theorem 11 will show, the general question has no simple answer, but we can settle the case when I is principal. The case $J_a = (\pi)_a$ is simpler. As noticed above, a necessary condition is that (π) is am-closed and this requires π to be regular, that is, (π) to be stable. The condition is also sufficient. Indeed, if $J_a = (\pi)_a = (\pi)$, then $J^- = (\pi)$ is principal and hence J too is principal and so it must coincide with (π). The ${}_aJ = {}_a(\pi)$ case has the same answer but its proof requires the use of lattice properties of principal ideals and Theorem 10. In summary:

Theorem 11.

(*i*) $J_a = (\pi)_a$ implies $J = (\pi)$ if and only if (π) is stable.
(*ii*) ${}_aJ = {}_a(\pi)$ implies $J = (\pi)$ if and only if (π) is stable.

For general ideals, the stability of I is no longer sufficient in either case and we find the counterexamples interesting. For case (*i*) we construct an ideal L which is not stable but whose arithmetic mean L_a is stable. L is generated by sequences $\pi(n)$ chosen so that for each n, $\omega_{a^n} \leq \pi(n)_a \leq \omega_{a^{2n}}$, but such that diag $\omega \notin L$. Then $L_a = \cup(\omega)_{a^n}$ is the *am-stabilizer* of (ω) and hence it is stable, but diag $\omega \in L_a \setminus L$ so L is not. For case (ii), we take $I = \cup(\omega)_{a^n}$ and set $J = I + (\pi)$, where π is chosen so that diag $\pi \notin I$ and hence $J \neq I$, but for each n, $(\omega_{a^n}) = (\omega_{a^n} + \pi)^o$ so ${}_aJ = {}_aI$.

Directly from the definition of am-closure (respectively, am-interior) it follows that I is am-closed (respectively, am-open) precisely when $J_a \subset I_a$ implies $J \subset I$. (respectively, ${}_aJ \supset {}_aI$ implies $J \supset I$). The reverse direction is subtler. It turns out that $(\omega^{1/2})_a \subset I_a$ does not imply that $(\omega^{1/2}) \subset I$ but only that $(\omega) \subset I$. In fact, (ω) is the largest ideal with this property. More generally, if $0 < p < 1$ and if $1/p - 1/p' = 1$, then $(\omega^{p'})$ is the largest ideal J such that $(\omega^p) \subset I_a$ implies $J \subset I$. We can generalize this result to all principal ideals (π). If π is summable then for every proper ideal I, $I_a \supset (\omega) = (\pi)_a$. Therefore $\cap\ \{J | J_a \supset (\pi)_a\}$ is the ideal F of finite rank operators which is principal and is generated by $\pi^\sim := (1, 0, 0, \ldots)$. If π is nonsummable, set $\phi_n := \min\ \{k | \Sigma_{i=1}^k \pi_i \geq n\}$ and define $\pi_n^\sim := (\pi_a)_{\phi_n}$. Then we prove that $(\pi)_a \subset I_a$ implies $(\pi^\sim) \subset I$ and that $(\pi^\sim)$ is the largest ideal with that property, namely:

Proposition 12. For all $\pi \in c_o^*$, $(\pi^\sim) = \cap\{J | J_a \supset (\pi)_a\}$.

In particular, $J_a \supset (\pi)_a$ implies $J \supset (\pi)$ if and only if $(\pi) = (\pi^\sim)$. We have an example of a regular sequence π satisfying the latter condition, but as mentioned above, for $0 < p < 1$, the regular sequences ω^p do not.

The situation for the pre-arithmetic mean ideal ${}_aI$ is different: for every π there exists ξ such that ${}_a(\xi) \subset {}_a(\pi)$ but $(\xi) \not\subset (\pi)$.

Second Order Cancellation

Wodzicki asked whether or not $(\xi)_{a^2} = (\pi)_{a^2}$ implies $(\xi)_a = (\pi)_a$. If π is regular, then the answer is clearly affirmative. We found that in general the answer is negative and then we investigated the properties of π which guarantee that this cancellation holds. This led to properties of the ratio $h(\pi_a) = \pi_{a^2}/\pi_a$. The first step is Proposition 13.

Proposition 13. Given an ideal I, then $J_{a^2} \subset I_{a^2}$ implies $J_a \subset I_a$ if and only if $I_a = (I_a)^{-o}$. A sufficient condition is that each $\pi \in \Sigma(I)$ is dominated by some $\xi \in \Sigma(I)$ such that $h(\xi_a)$ is equivalent to a monotone nondecreasing sequence.

Notice that for all $\xi \in c_o^*$, $1 \leq h(\xi_a) \leq \log$ and that $\log - h(\xi_a)$ is strictly increasing (to infinity if ξ is nonsummable). So we have two extremal cases: when $h(\xi_a) \asymp 1$, i.e., ξ_a and hence ξ are regular, and when $h(\xi_a) \asymp \log$. Interestingly, it turns out that the latter case is equivalent to $\sigma(\xi)$ satisfying the *exponential* Δ_2-*condition*, i.e., for some $c > 0$, $\Sigma_{i=1}^{n^2} \xi_i \leq c\ \Sigma_{i=1}^{n} \xi_i$ for all n. This condition is also sufficient for second order cancellation for the reverse inclusions.

Proposition 14. Let I be an ideal such that every $\pi \in \Sigma(I)$ is dominated by some $\xi \in \Sigma(I)$, such that $\sigma(\xi)$ satisfies the exponential Δ_2-condition. Then $J_{a^2} \supset I_{a^2}$ implies $J_a \supset I_a$.

Notice that this cancellation can fail even for I principal and stable, e.g., for $I = (\omega^{1/2})$. Referring back to Wodzicki's question we see that $(\xi)_{a^2} = (\pi)_{a^2}$ implies $(\xi)_a = (\pi)_a$ in the cases when π is regular or when $\sigma(\pi)$ satisfies the exponential Δ_2-condition (the two extreme cases). But in general the answer is negative. The counterexample outlined below illustrates some of the features of the theory.

There is an increasing sequence n_k and two c_o^*-sequences $\eta \geq \xi$ defined by $\xi_j := e^{-k}\eta_{n_k}$ for $n_k < j \leq n_{k+1}$ and $\eta_j := \sqrt{k}\, e^{-k}\eta_{n_k}$ for $n_k < j \leq e^k n_k$ and $\eta_j = \xi_j$ for $e^k n_k < j \leq n_{k+1}$. The n_k are taken

MATHEMATICS

sufficiently large to achieve the asymptotics $(\xi)_{n_k} \sim (\xi_a)_{n_k} \sim (\xi_{a^2})_{n_k} \sim \eta_{n_k} \sim (\eta_a)_{n_k} \sim (\eta_{a^2})_{n_k}$ (a "resetting of the clock" process). The ratio η_a/ξ_a increases throughout the interval $n_k < j \le e^k n_k$, becomes "large" for $j \approx e^k n_k$, and then decreases. The ratio η_{a^2}/ξ_{a^2} is more "resistant to change." It increases more slowly but continues to increase past $e^k n_k$ and reaches a maximum on $e^k n_k < j \le n_{k+1}$. Fine-tuning of the growth constants so to make the ratio η_{a^2}/ξ_{a^2} close to 1 for $j \approx e^k n_k$ guarantees that this maximum can be kept uniformly bounded. Consequently, $\eta_a \not\asymp \xi_a$ but $\eta_{a^2} \asymp \xi_{a^2}$.

1. Calkin, J. W. (1941) *Ann. Math.* **42,** 839–873.
2. Halmos, P. R. (1954) *Am. J. Math.* **76,** 191–198.
3. Pearcy, C. & Topping, D. (1971) *Michigan J. Math.* **18,** 247–252.
4. Dykema, K., Figiel, T., Weiss, G. & Wodzicki, M. (2001) http://www.msri.org/publications/preprints/online/2001-013.html.
5. Weiss, G. (1980) *Integral Equations and Operator Theory* **3/4,** 574–600.
6. Kalton, N. J. (1989) *J. Funct. Anal.* **86,** 41–74.
7. Dixmier, J. (1966) *C. R. Acad. Sci. Paris Ser. A-B* **262,** A1107–A1108.
8. Macaev, V. I. (1961) *Dokl. Akad. Nauk SSSR* **139,** 548–551.
9. Connes, A. (1994) *Noncommutative Geometry* (Academic, New York).
10. Gohberg, I. C. & Krein, M. G. (1969) *Introduction to the Theory of Linear Non-selfadjoint Operators*, Translations of Mathematical Monographs, trans. Feinstein, A. (Am. Math. Soc., Providence, RI), Vol. 18.
11. Varga, J. V. (1989) *Proc. Amer. Math. Soc.* **107,** 715–723.
12. Fuglede, B. (1950) *Proc. Nat. Acad. Sci. USA* **36,** 35–40.
13. Putnam, C. R. (1951) *Am. J. Math.* **73,** 357–362.
14. Weiss, G. (1983) *Trans. Am. Math. Soc.* **278,** 1–20.
15. Voiculescu, D. (1979) *J. Operator Theory* **2,** 3–37.
16. Shulman, V. S. (1983) *Soviet Math. Dokl.* **27,** 726–729.
17. Dykema, K., Weiss, G. & Wodzicki, M. (2000) in *Complex Analysis and Related Topics, Cuernavaca, 1996* (Birkhauser, Basel), pp. 59–65.
18. Kalton, N. J. (1987) *Math. Nachr.* **134,** 119–130.
19. Blass, A. & Weiss, G. (1978) *Trans. Amer. Math. Soc.* **246,** 407–417.