Supporting Information

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SI Text

Proof of Lemma 1

From the law of motion at the end of the discrete-time model, it follows that

$$q_h(t) - q_h(t-h) = -khq_h(t-h) + kh\left(1 - F\left[\frac{p(t)}{q_E(t,h)}\right]\right).$$

Dividing both sides by h, and letting h tend to zero, gives the desired result.

Proof of Theorem 1

Let (a^{**}, q^{**}) be a rational expectations equilibrium (REE). Because $v(0) = 0$, the quantity q^{**} is strictly positive. We now demonstrate the existence of a price trajectory that, starting with the initial condition

$$q(0) = q^{**},$$

in the model of boundedly rational myopic consumers, yields a profit that is strictly greater than $(1/r)v(q^{**})$. Let $u > 0$, and define the pricing trajectory $\boldsymbol{a}$ by

$$\boldsymbol{a}(t) = \begin{cases} 1, & 0 \le t < u, \\ A[q(u)], & t \ge u. \end{cases} \quad \textbf{[S1]}$$

The total discounted profit from $\boldsymbol{a}$ is

$$w(u) = \int_0^u e^{-rx}[q(x)]^2 dx + \frac{e^{-ru}v[q(u)]}{r}, \quad u \ge 0,$$

$$v[q(u)] = A[q(u)][q(u)]^2. \quad \textbf{[S2]}$$

Differentiating w, one gets

$$w'(u) = e^{-ru}\left\{[q(u)]^2 - v[q(u)] + \left(\frac{1}{r}\right)v'[q(u)]q'(u)\right\}. \quad \textbf{[S3]}$$

Setting $u = 0$ one gets

$$w'(0) = (q^{**})^2 - v(q^{**}) + \left(\frac{1}{r}\right)v'(q^{**})q'(0) \quad \textbf{[S4]}$$

$$= (q^{**})^2 - a^{**}(q^{**})^2 + 0 \quad \textbf{[S5]}$$

$$= (q^{**})^2(1 - a^{**}) \quad \textbf{[S6]}$$

$$> 0. \quad \textbf{[S7]}$$

But

$$w(0) = \left(\frac{1}{r}\right)v(q^{**}). \quad \textbf{[S8]}$$

Hence, for some sufficiently small $\varepsilon > 0$,

$$w(\epsilon) > w(0), \quad \textbf{[S9]}$$

so that setting $u = \varepsilon$ yields the monopolist a larger total discounted profit than charging the REE price forever from time 0, which completes the proof of *Theorem 1*.

Detailed Discussion of the Case of a Convex Distribution of Consumer Types: Proofs of Theorems 2 and 3

In this section we consider the case where the cumulative distribution function of consumer types is strictly convex: $F''(a) > 0$, $0 < a < 1$. We present the proofs of *Theorems 2* and *3* and illustrate these results with the quadratic cumulative distribution function (cdf) $F(a) = a^2$.

In this section, we use the notation $\dot{x}$ to denote the derivative with respect to time, whereas a "prime" denotes the derivative with respect to the function's argument:

$$\dot{F}(a(t)) = \frac{dF(a(t))}{dt}, \quad F'(a) = \frac{dF}{da}.$$

We use notation in boldface type to denote a function; for example, $\boldsymbol{a}$ is the entire trajectory of the control variable for $t \ge 0$; whereas a variable that is the value of a function evaluated at a particular point is set in regular type; for example, $a(t)$ is the value of the function (trajectory) $\boldsymbol{a}$ at the particular time t. When there is no danger of confusion between the two, we sometimes omit the boldface type for legibility.

Recall that the monopolist wants to solve the following optimization problem:

$$\max_{\boldsymbol{a}} = \int_0^\infty e^{-rt}a(t)q(t)^2 dt = \int_0^\infty e^{-rt}h(q(t), a(t))\,dt, \quad \textbf{[S10]}$$

$$\text{subject to } \dot{q}(t) = k[1 - F(a(t)) - q(t)] = m[q(t), a(t)], \quad \textbf{[S11]}$$

$$q(0) = q_0 \in (0,1), \quad 0 \le a \le 1. \quad \textbf{[S12]}$$

Our assumptions on F imply that $0 \le F \le 1$ for all a, and $F'(a) \ge 0$ and $F''(a) > 0$ for all $a \in (0, 1)$. Also note that, with these constraints on F, if $q_0 \in [0, 1]$, then $q(t)$ will remain between 0 and 1.

We use an infinite-time maximum principle proved by Weber in ref. 1, proposition 2. Although this problem does not satisfy all of the hypotheses of the cited proposition, the conclusion still holds, and the proof—given below—closely follows the argument in that paper. For the optimization problem **[S10]**, define the current value Hamiltonian function

$$H(q, a, \psi) = aq^2 + k\psi[1 - F(a) - q],$$

where ψ is the adjoint, or costate, variable. We denote by $\mathcal{A}$ the set of admissible controls, in this case those that are bounded between 0 and 1.

Proposition 1: Pontryagin Maximum Principle. *Let* $(\boldsymbol{q}^*, \boldsymbol{a}^*)$ *be a solution to the optimal control problem* **[S10]**. *Then there exists an absolutely continuous function* $\boldsymbol{\psi}$: $\mathbb{R}^+ \to \mathbb{R}$ *such that the following optimality conditions are satisfied*:

i) *Adjoint equation*

$$\dot{\psi}(t)=r\psi(t)-\frac{\partial H}{\partial q}\left[q^{*}(t),a^{*}(t),\psi(t)\right], \quad \textbf{[S13]}$$

ii) *Maximality condition*

$$H\left[q^{*}(t),a^{*}(t),\psi(t)\right]=\sup_{0\le a\le 1}H\left[q^{*}(t),a,\psi(t)\right], \quad \textbf{[S14]}$$

iii) *Boundedness of the adjoint variable. The function* ψ *is positive and bounded on* $\mathbb{R}^{+}$. *In particular, there exists a number* $\overline{\psi}$ *such that*

$$0\le\psi(t)\le\overline{\psi}$$

for all $t\in\mathbb{R}^{+}$.

Note that if $\boldsymbol{a}^{*}$ satisfies

$$\frac{\partial H}{\partial a}\left[q^{*}(t),a^{*}(t),\psi(t)\right]=0, \quad \textbf{[S15]}$$

then it is a candidate solution for Eq. **S14**.

Here we include an outline of the proof of *Proposition 1*, adapted from ref. 1 to the particular constraints of our problem. We sketch here the main arguments of the proof, with details only where modifications are needed for our particular application.

Outline of Proof of Proposition 1. We break the proof up into steps. Denote by (P) the infinite-horizon problem given in Eqs. **S10** and **S11**. In what follows, it is more convenient to use $u=F^{-1}(a)$ as the control variable; we apologize for the clumsiness in notation.

Existence of a Solution. The state space, Q, for the problem is the interval [0, 1]. As remarked earlier, the assumptions on F imply that Q is invariant under the law of motion and so admissible trajectories are uniformly bounded between 0 and 1. We also note that, for all $T>0$,

$$\omega(T)=\int_{T}^{\infty}e^{-rt}G(u(t))q(t)^{2}dt\le\int_{T}^{\infty}e^{-rt}dt=\frac{e^{-rT}}{r}$$

defines a nonincreasing positive function $\omega(T)$ with $\lim_{T\to\infty}\omega(T)=0$.

The current value of the integrand in the value function $h(u,q)=G(u)q^{2}$ satisfies

$$\frac{\partial^{2}h}{\partial u^{2}}=q^{2}G''(u)<0$$

because $F=G^{-1}$ is convex, so h is strictly concave with respect to u. This concavity ensures the uniqueness of the optimal u^{*} in terms of q and the costate variable, ψ. Theorem 3.6 in ref. 2 can then be applied to ensure the existence of a solution to the infinite-horizon problem (P).

Construction of the Solution. As in ref. 1 we define a sequence of finite-time optimization problems, P_n, on time intervals $[0, T_n]$, where $T_n<T_{n+1}$ and $\lim_{n\to\infty}T_n\to\infty$. In our case, the control u is bounded by 1, so that $1+|u|\le 2$ for all controls. Denote by (q^{*}, u^{*}) an optimal pair for the infinite-horizon problem (P). We require a sequence $v_n\in C^{1}(\mathbb{R}_{+},\mathbb{R})$ and an increasing sequence of real constants $\varepsilon_n\to\infty$ that satisfy

$$\sup_{t\in\mathbb{R}_{+}}|v_n(t)|\le 2,\qquad \int_{0}^{\infty}e^{-rt}|v_n-u^{*}|^{2}dt\le\frac{1}{n}$$

and

$$\frac{1}{n(1+\epsilon_n)}\ge\omega(T_n).$$

Such sequences exist because $\{v_n\}$ can be obtained by successive approximations of u^{*} and $\{\varepsilon_n\}$ by approximating the sequence $\frac{1}{n\cdot\omega(T_n)}$, which diverges. We can then define the sequence of optimization problems

$$(P_n)\ \text{Maximize}\quad V(u)=\int_{0}^{\infty}e^{-rt}\left(G(u)q^{2}-\frac{|v_n-u|^{2}}{1+\epsilon_n}\right)dt,$$

subject to the same constraints as the problem (P). It can be shown (1) that there is an optimal u_n for each problem (P_n) and that, for any $T>0$, these optimal solutions converge in L_2 to the optimal u^{*} for problem (P) as $n\to\infty$. As a consequence of this L_2 convergence, we can conclude that, for any $T>0$ as $n\to\infty$,

$$u_n\to u^{*}\quad\text{in}\quad L_2[0,T], \quad \textbf{[S16]}$$

$$q_n\rightrightarrows q^{*}\quad\text{on}\quad[0,T], \quad \textbf{[S17]}$$

$$\dot{q}_n\to\dot{q}^{*}\quad\text{weakly in}\quad L_1[0,T]. \quad \textbf{[S18]}$$

See ref. 1 for details and further references.

By the classical Pontryagin maximum principle (3) there is an absolutely continuous adjoint variable, ψ_n for problem P_n, i.e., one that satisfies the adjoint equation, Eq. **S13**, with

$$H_n(q_n,u_n,\psi_n)\overset{a.e.}{=}\sup_{u}\mathcal{H}_n(q_n,u,\psi_n)=\hat{H}_n(q_n,\psi_n),$$

where

$$\hat{H}_n=G(u_n)q_n^{2}-\frac{|v_n-u_n|^{2}}{1+\epsilon_n}+\psi_n(-ku_n+k(1-q_n)).$$

Then we can show the following:

Claim 1. *$|\psi_n(0)|$ is bounded for large n.*

Proof: We drop the subscripts, n, for legibility. Using the Pontryagin maximum principle for the finite-horizon problem, the adjoint equation is

$$\dot{\psi}=(r+k)\psi-2G(u)q.$$

This has the solution

$$\psi(T)e^{-(r+k)T}-\psi(0)=-2\int_{0}^{T}e^{-(r+k)t}G(u)q(t)dt. \quad \textbf{[S19]}$$

For the finite-time problem, the transversality condition implies that $\psi(T)=0$, which implies that

$$\psi(0)=2\int_{0}^{T}e^{-(r+k)t}G(u)q(t)dt.$$

Because $0 \le G(u) \le 1$ and $0 \le q \le 1$, we get the following bounds for $\psi(0)$:

$$0 \le \psi(0) \le 2\int_0^T e^{-(r+k)t}\,dt = \frac{2}{r+k}\left(1-e^{-(r+k)T}\right) < \frac{2}{r+k}.$$

Because this will hold for all finite times, T, we have that $|\psi_n(0)|$ is bounded by $\frac{2}{r+k}$ independent of n, which proves the claim.

□

An identical argument to that given in ref. 1 about the convergence properties of the solutions to the finite-time problems as $T_n \to \infty$ shows that there is a sequence of adjoint solutions ψ_n that converge uniformly on every interval $[0, T_n]$, to an adjoint solution, ψ, of the infinite-time problem. Furthermore $\dot{\psi}_n \to \dot{\psi}$ weakly in $L_1([0, T_n])$. We see that $\psi(t)$ is positive because $\psi_n > 0$ on $[0, T_n)$ for $n \to \infty$. The upper bound on $|\psi_n(0)|$ is also an upper bound of $|\psi_n(t)|$ for $t \in [0, T_n]$ for all n: this gives the bounds on the adjoint variable, ψ in *Proposition 1*.

This completes the *Proof* of *Proposition 1*.

□

In this section we use *Proposition 1* to find differential equations for the adjoint, $\boldsymbol{\psi}$, and the optimal control, $\boldsymbol{a}^*$, and we use the structure of the phase space of these functions and the boundedness of $\boldsymbol{\psi}$ to show that the optimal trajectories must converge to a unique equilibrium. Furthermore, using the geometry of the phase space, we show that, for every initial value of the demand, q_0, there is a unique optimal trajectory $(\boldsymbol{q}^*, \boldsymbol{a}^*)$ through q_0.

Uniqueness of the Optimal Strategy, Proof of Theorem 2. In this section, we analyze the system in the $q\psi$ plane to show that any optimal trajectory must lie on a curve. From this we conclude that, for each value of $q(t)$, there is a unique value $\psi(t)$ that corresponds to an optimal trajectory and hence, by Pontryagin's maximum principle, a unique optimal policy $a(t) = \alpha(q(t))$.

Our strategy is to describe all possible long-term behaviors of the system by first finding all possible equilibria and their stability. This is done by showing that the nullclines of the system intersect each other exactly twice, once at the origin and once in the interior of the unit square. Thus, there are two equilibria of the system, one at (0, 0) and one in the interior. Fig. S1 shows an example of the phase portrait. The Poincaré–Bendixson theorem tells us that bounded orbits must converge to an equilibrium, a cycle graph, or a limit cycle (theorem 9.2.1 in ref. 4). A linear analysis, given in *Lemma 1*, shows that the interior equilibrium is a saddle. Cycles can be ruled out using Bendixson's criterion (theorem 9.2.4 in ref. 4) or index theory, and therefore the long-term behavior of all trajectories that start in the unit square must fall into one of three categories:

i) The trajectory is unbounded.
ii) The trajectory converges to the origin.
iii) The trajectory converges to the interior saddle equilibrium along its stable manifold.

The first category is clearly not optimal. *Lemma 2* shows that the second category is not optimal. Therefore, the unique optimal strategy is the one that corresponds to a trajectory along the stable manifold of the saddle.

(Note that henceforth, to simplify the notation, we no longer use boldface type to distinguish functions from scalar variables.)

For clarity in the calculations, we let $u = F(a)$, so that $G(u) = F^{-1}(u) = a$. The equations of motion, Eqs. **S11** and **S13**, become

$$\dot{\psi} = (r+k)\psi - 2G(u)q, \qquad \textbf{[S20]}$$

$$\dot{q} = k(1-u-q). \qquad \textbf{[S21]}$$

The main argument in the proof of *Theorem 2* is contained in the following:

Lemma 1. *The system of differential equations given by Eqs.* **S20** *and* **S21** *has a unique equilibrium in the interior of the unit square. This interior equilibrium is a saddle.*

Proof: Along optimal trajectories, the interior maximality condition, Eq. **S15**, implies that

$$g(u) = G'(u) = \frac{k\psi}{q^2} \Rightarrow u = g^{-1}\left(\frac{k\psi}{q^2}\right). \qquad \textbf{[S22]}$$

To find the equilibria we calculate the nullclines by setting Eqs. **S20** and **S21** equal to 0. We can write u in terms of q and ψ, using Eq. **S22**, whence, after substitution into the nullcline equations, we get ψ in terms of q only:

$$\dot{q} = 0 \Rightarrow \psi = N_1(q) = \frac{q^2 g(1-q)}{k}, \qquad \textbf{[S23]}$$

$$\dot{\psi} = 0 \Rightarrow \psi = N_2(q) = \frac{q^2}{k} g\left(\lambda^{-1}\left(\frac{2k}{(r+k)q}\right)\right) \qquad \textbf{[S24]}$$

for $q \ne 0$, where $L(u) = \ln G(u)$, and $\lambda(u) = L'(u) = \frac{g(u)}{G(u)}$. Because we assume that F is strictly increasing on [0, 1], we can define $G(1) = F^{-1}(1) = 1$. However, the derivative of G is not well defined at $u(t) = 1$ or $u(t) = 0$, so the nullcline equations are valid only on the open interval (0, 1).

At an equilibrium where $q \ne 0$, we must have

$$q = 1 - u = \left(\frac{2k}{r+k}\right)\frac{1}{\lambda(u)}.$$

Because $F'(a) > 0$ and increasing, we have $g(u) = G'(u) > 0$ and decreasing, so that $\lambda(u) = \frac{g(u)}{G(u)} > 0$ and decreasing as well, which means that $\frac{1}{\lambda(u)}$ is increasing, so that there is only one equilibrium at which $q \ne 0$.

We consider the case $q = 0$ separately: If $\dot{q}(t) = 0$ when $q(t) = 0$, we must have $u(t) = 1 \Rightarrow G(u(t)) = 1$, so that we could have an equilibrium where $q = 0$ only if $\psi = \frac{2q}{r+k} = 0$.

Note that $\lim_{q\to 1} N_1(q) = \lim_{u\to 0} \frac{g(u)}{k}$. On the other hand,

$$\lim_{q\to 1} N_2(q) = \lim_{q\to 1} \frac{1}{k} g\left(\lambda^{-1}\left(\frac{2k}{(r+k)q}\right)\right).$$

Thus, because g is a decreasing function, as long as $\lambda^{-1}\left(\frac{2k}{(r+k)q}\right) > 0$ (which holds if F is strictly convex), we have

$$\lim_{q\to 1} N_1(q) > \lim_{q\to 1} N_2(q).$$

On the other hand, if $g(u)$ is bounded as $u \to 1$, then, from the definition of λ, we know that $\lim_{u\to 0} \lambda(u) \to \infty$ so that we get

$$\lim_{q\to 0} N_1(q) = \lim_{q\to 0} N_2(q) = 0.$$

Therefore, the two graphs of the two nullclines can intersect at two equilibria: once in the interior of the unit square and once at the origin. [See Fig. S1 for the case $F(a) = a^2 \Rightarrow G(u) = \sqrt{u}$.] The stability of the interior equilibrium can be calculated by looking at the linearized system, given by the Jacobian matrix

$$J = \begin{bmatrix} r+k-2qG'(u)\dfrac{du}{d\psi} & -2G(u)-2qG'(u)\dfrac{du}{dq} \\ -k\dfrac{du}{d\psi} & -k\dfrac{du}{dq}-k \end{bmatrix}.$$

The derivatives of u with respect to q and ψ can be evaluated using Eq. **S22**:

$$\frac{du}{d\psi}=\frac{1}{g'(u)}\frac{k}{q^2}=\frac{g(u)}{g'(u)}\frac{1}{\psi},\quad \frac{du}{dq}=\frac{1}{g'(u)}\frac{-2k\psi}{q^3}=\frac{g(u)}{g'(u)}\frac{2}{q}. \qquad \textbf{[S25]}$$

The equilibrium is a saddle if the determinant of the Jacobian, evaluated at the equilibrium, is negative. Recall that $g = G'$:

$$\begin{aligned}\det(J) = -k\left(\left[(r+k)-2qg(u)\frac{du}{d\psi}\right]\left(\frac{du}{dq}+1\right)\right)\\ -k\left(2\left[G(u)+qg(u)\frac{du}{dq}\right]\frac{du}{d\psi}\right).\end{aligned} \qquad \textbf{[S26]}$$

At the equilibrium, we know that $q=1-u=\frac{2k}{r+k}\frac{g(u)}{G(u)}$ and $(r+k)\psi = 2G(u)q$. Use these identities and Eqs. **S25** to get

$$\frac{du}{d\psi}=\frac{(r+k)^2}{4kg'(u)},\qquad \frac{du}{dq}=-\frac{(r+k)G(u)}{kg'(u)},$$

which gives, after simplification,

$$\det(J)=-k\left[\frac{-(r+k)^2G(u)}{2kg'(u)}+\frac{-(r+k)g(u)^2}{2G(u)g'(u)}+(r+k)\right]. \qquad \textbf{[S27]}$$

By our assumptions on F, $G(u)$ and $g(u)$ are positive, whereas $g'(u)$ is negative, so that all terms inside the parentheses are positive, and therefore $\det(J)$ is negative, which implies that the interior equilibrium is a saddle.

Because the interior equilibrium is a saddle, the only trajectories that converge to this equilibrium lie on its stable manifold. All other trajectories either are unbounded or converge to $q = \psi = 0$. The former is not optimal by the boundedness of the adjoint variable (condition *iii* of *Proposition 1*). *Lemma 2* excludes trajectories that converge to the origin.

Lemma 2. *Trajectories in which* $q \to 0$ *are not optimal.*

Proof: For a fixed discount rate, r, and parameter k, define $\epsilon=\frac{1}{3+r}\left(\frac{1+r}{3+r}\right)^{\frac{r+k}{2k}}$ and $\delta=\frac{2}{3+r}$. Consider $\boldsymbol{\alpha_0}$, a policy such that $q(t)\to 0$. Let T be a time such that, under the policy $\boldsymbol{\alpha_0}$, $q(t)<\epsilon, \forall t>T$. The value of this policy can be written as

$$\begin{aligned}V_{\boldsymbol{\alpha_0}}&=\int_0^\infty e^{-rt}a(t)q^2(t)\,dt<\int_0^T e^{-rt}a(t)q^2(t)\,dt+\int_T^\infty e^{-rt}\epsilon^2\,dt\\ &=V_T+\frac{\epsilon^2}{r}e^{-rT}.\end{aligned} \qquad \textbf{[S28]}$$

We show that $\boldsymbol{\alpha_0}$ is not optimal by defining a policy whose value is larger than $V_{\boldsymbol{\alpha_0}}$. Suppose that at time T, a is set equal to 0. Then the time derivative of q is positive, because $\dot q = k(1-F(a)-q) = k(1-q)>0$, and therefore $q(t)$ will increase. In fact, with this choice of $a(t)$, we get

$$q(t)=1-e^{-k(t-T)}[1-q(T)],\qquad t>T. \qquad \textbf{[S29]}$$

Thus, there is a time, T_δ such that $q(T_\delta) = \delta$. Consider the policy, $\hat{\boldsymbol{\alpha}}$ given by

$$\hat{\boldsymbol{\alpha}}(t)=\begin{cases}\boldsymbol{\alpha_0}(t) & t\le T\\ 0 & T<t\le T_\delta\\ F^{-1}(1-\delta) & T_\delta<t\end{cases}.$$

Note that this policy ensures that $q(t) = \delta$ for all $t > T_\delta$. Let $V_{\hat\alpha}$ be the value of this new policy. Then, from Eq. **S28** we have

$$\begin{aligned}V_{\hat{\boldsymbol{\alpha}}}-V_{\boldsymbol{\alpha_0}}&>\left(V_T+\frac{\delta^2F^{-1}(1-\delta)}{r}e^{-rT_\delta}\right)-\left(V_T+\frac{\epsilon^2}{r}e^{-rT}\right)\\ &=\frac{1}{r}\left[\delta^2F^{-1}(1-\delta)e^{-rT_\delta}-\epsilon^2e^{-rT}\right] \qquad \textbf{[S30]}\\ &\ge\frac{1}{r}e^{-rT}\left[\delta^2F^{-1}(1-\delta)(1-\delta)^{r/k}-\epsilon^2\right]\\ &>\frac{1}{r}e^{-rT}\left[\delta^2(1-\delta)^{r/k+1}-\epsilon^2\right], \qquad \textbf{[S31]}\end{aligned}$$

where the inequality in Eq. **S30** is derived by solving for $q(t) = \delta$ in Eq. **S29**, and Eq. **S31** follows from the fact that F is a strictly convex cdf. Now the choice of ε and δ gives

$$\delta^2(1-\delta)^{r/k+1}=4\left(\frac{1}{3+r}\right)^2\left(\frac{r+1}{3+r}\right)^{r/k+1}>\epsilon^2,$$

so that the we have $V_{\hat{\boldsymbol{\alpha}}}-V_{\boldsymbol{\alpha_0}}>0$ and therefore $\boldsymbol{\alpha_0}$ cannot be optimal. □

By *Lemma 1*, all trajectories of the system of differential equations are unbounded, go to zero, or lie on the stable manifold of the interior equilibrium. Unbounded trajectories cannot be optimal, and *Lemma 2* shows that trajectories with $q(t) \to 0$ are not optimal. Therefore, all optimal trajectories must lie on the stable manifold of the interior equilibrium. This stable manifold can be broken into two pieces: one with q values below the interior equilibrium and one with q values above the interior equilibrium. By construction, neither of these pieces can cross a nullcline, and therefore both q and ψ are either strictly increasing or strictly decreasing along each segment. Thus, the stable manifold of the interior saddle equilibrium implicitly defines a one-to-one mapping from q to ψ. A phase portrait for the case $F(a) = a^2$ is shown in Fig. S1.

Proof of Theorem 2: Recall that we are assuming that F is strictly convex and that $(\boldsymbol{q}^*, \boldsymbol{a}^*)$ is a solution to the optimal control problem [**S10**]. Eqs. **S13**, **S15**, and **S11**, respectively, give the following for the optimal triple $(\boldsymbol{q}^*, \boldsymbol{a}^*, \boldsymbol{\psi}^*)$:

$$\dot\psi=(r+k)\psi-2aq, \qquad \textbf{[S32]}$$

$$0=q^2-k\psi F'(a), \qquad \textbf{[S33]}$$

$$\dot q=k(1-F(a)-q). \qquad \textbf{[S34]}$$

We use these to get an equation for the time evolution of the optimal $a(t)$. The maximality condition, Eq. **S33**, gives

$$F'(a)=\frac{q^2}{k\psi}\Rightarrow\psi=\frac{q^2}{kF'(a)}. \qquad \textbf{[S35]}$$

Differentiation of Eq. **S35** with respect to time and substituting expressions for $\dot q$, ψ, and $\dot\psi$ yields

$$\begin{aligned}F''(a)\dot a&=\frac{2q\dot q\psi-q^2\dot\psi}{k\psi^2}\Rightarrow\\ \dot a&=\frac{F'(a)}{qF''(a)}\left[2k(1-F(a)-q)-(r+k)q+2kaF'(a)\right].\end{aligned} \qquad \textbf{[S36]}$$

Eqs. **S11** and **S36** give two differential equations for $q(t)$ and $a(t)$ along an optimal trajectory. We show directly that this system has a unique equilibrium in the interior of the unit square.

Any equilibria of the system will lie on the intersection of the q and a nullclines, given by

$$\dot{q}=0 \Rightarrow q=N_1(a)=1-F(a),$$
$$\dot{a}=0 \Rightarrow F'(a)=0, \text{ or } q=N_2(a)=\frac{2k[1-F(a)+aF'(a)]}{r+3k}. \quad \textbf{[S37]}$$

We remark that the assumptions on F imply that $F'(a) > 0$ so that N_1 is strictly decreasing and concave for $a \in (0, 1)$, and $N_1(0) = 1$, $N_1(1) = 0$.

For the other nullcline, we calculate that $N_2(0)=\frac{2k}{r+3k}<1$, and the derivative of $N_2(a)$ is always positive,

$$\frac{dN_2}{da}=\frac{2k}{(r+3k)}\left[-F'(a)+F'(a)+F''(a)\right]>0$$

because $F''(a) > 0$. This means that the graph of $N_2(a)$ starts below the graph of N_1 at $a = 0$ and ends up above the graph of N_1 at $a = 1$, and the two graphs intersect exactly once, because N_2 is strictly increasing whereas N_1 is strictly decreasing. Therefore, there is exactly one equilibrium where the graphs of $N_1(a)$ and $N_2(a)$ intersect.

If $F'(0) = 0$, there is another equilibrium at $a = 0$, $q = 1$. Note that, if $F'(0) = 0$, trajectories in the qa plane that converge to the equilibrium at $q = 1$, $a = 0$ correspond to trajectories in the ψq plane with $\psi \to \infty$, because

$$a \to 0 \Rightarrow g(u) \to \frac{1}{F'(0)} \Rightarrow \psi \to \frac{1}{kF'(0)} \to \infty.$$

By the boundedness of ψ on optimal trajectories (*Proposition 1*, condition *iii*), trajectories in the qa plane that converge to $(a, q) = (0, 1)$ are not optimal. Therefore, as in the previous section, optimal trajectories must lie on the stable manifold of the interior equilibrium.

The phase plane of the system, with nullclines depicted and arrows indicating the direction of the vector field, is given in Fig. S2. For this example, we have taken $F(a) = a^2$, so that we are in the case $F'(0) = 0$.

Let (q^{**}, a^{**}) denote the coordinates of this interior equilibrium, and denote by $\mathcal{W}_S$ the stable manifold of the saddle. Let (q_1, a_1) be a point on $\mathcal{W}_S$ with $q_1 < q^{**}$. Because this portion of $\mathcal{W}_S$ lies above N_2 and below N_1, the vector field at (q_1, a_1) is in the first quadrant, i.e., $\dot{q}(t)>0$ and $\dot{a}(t)>0$, so that $q(t)$ and $a(t)$ increase monotonically toward (q^{**}, a^{**}). In other words, $\mathcal{W}_S$ is strictly increasing in this region (as a function of q) so that, for every initial $q_1 < q^{**}$ there is a unique optimal strategy starting at a_1, and both $q(t)$ and $a(t)$ will increase monotonically over time. A similar argument shows that for an initial $q_2 > q^{**}$ there is a unique optimal strategy starting at a_2 on $\mathcal{W}_S$, and both $q(t)$ and $a(t)$ decrease monotonically toward (q^{**}, a^{**}) from this value.

By Pontryagin's maximum principle, *Proposition 1*, this phase portrait gives all candidates for optimal trajectories. To determine which of the orbits given by this phase portrait are optimal, we appeal to *Lemmas 1* and *2* to conclude that any optimal trajectory must lie on the stable manifold of the interior saddle equilibrium.

We can interpret this result as follows: $\mathcal{W}_S$ determines a one-to-one mapping from values of q to strategies that give the stationary policy, $\alpha(q)=a$ analogous to that described for the case of the uniform distribution. Here we see that α is an increasing function of q. Also, as in the uniform case, $\alpha(q^{**})=a^{**}$ is the policy such that $\dot{q}=0$, which corresponds to the target in *Theorem 4*. The target is given analytically as the value (q^{**}, a^{**}) such that

$$q^{**}=1-F\left(a^{**}\right)=\frac{2k}{r+3k}\left[1-F\left(a^{**}\right)+a^{**}F'\left(a^{**}\right)\right]$$
$$\Rightarrow a^{**}=\frac{r+k}{2kF'\left(a^{**}\right)}. \quad \textbf{[S38]}$$

□

Some typical strategies and demand trajectories are shown in Fig. S3.

The REE Demand Is Greater Than the Target Demand in the Case of a Convex cdf, Proof of Theorem 3. The equation for the monopolist's profit is

$$v(q)=A(q)q^2=F^{-1}(1-q)q^2.$$

For a regular cdf, F, the REE is given by the value $\hat{q}$ that satisfies

$$\left.\frac{dv}{dq}(q)\right|_{\hat{q}}=0.$$

Letting $\hat{a}=F^{-1}(1-\hat{q})$ we get that the price at the REE must satisfy

$$v'(\hat{q})=0 \Rightarrow \frac{\hat{q}^2}{F'(F^{-1}(1-\hat{q}))(-1)}+2F^{-1}(1-\hat{q})\hat{q}=0,$$
$$\Rightarrow \hat{q}=0 \quad \text{or} \quad \hat{q}=2F^{-1}(1-\hat{q})F'(F^{-1}(1-\hat{q})),$$
$$\Rightarrow 2\hat{a}F'(\hat{a})=1-F(\hat{a}).$$

In the case of the convex cdf, the target demand is given in Eq. **S38**:

$$q^{**}=\frac{2k}{r+3k}\left[1-F(a^{**})+a^{**}F'(a^{**})\right]=1-F(a^{**}),$$
$$\Rightarrow 2a^{**}F'(a^{**})=\left[1-F(a^{**})\right]\left[\frac{r+k}{k}\right]=\left[1-F(a^{**})\right]\left[1+\frac{r}{k}\right].$$

We want to show that $a^{**}>\hat{a}$ so that $q^{**}<\hat{q}$; i.e., the target demand is less than the REE demand.

To show this, we note first that the function $aF'(a)$ is increasing in a, because its derivative $F'(a)+aF''(a)$ is strictly positive for $a\in(0,1)$. Therefore, because

$$[1-F(a)]<[1-F(a)]\left(1+\frac{r}{k}\right)$$

for all r, $k > 0$, we must have $\hat{a}<a^{**} \Rightarrow \hat{q}>q^{**}$, or the target demand is strictly less than the REE demand for $r > 0$, with equality when $r = 0$.

Example: The Quadratic Case. Here we illustrate *Theorems 2* and *3* by working through the details in the case of a quadratic distribution function:

$$F(a)=\begin{cases} a^2 & 0\le a\le 1 \\ 0 & a<0 \\ 1 & a>1 \end{cases}.$$

In the (q, a) variables, we have the following equations of motion:

$$\dot{a}=\frac{2a}{q}\left(2k(1-a^2-q)-(r+k)q+4ka^2\right) \quad \textbf{[S39]}$$

$$\dot{q}=k(1-a^2-q). \quad \textbf{[S40]}$$

The interior equilibrium is at

$$1-a^2=\frac{2k\left(1-a^2+2a^2\right)}{r+3k}$$
$$\Rightarrow\frac{r+3k}{2k}\left(1-a^2\right)=1+a^2\Rightarrow a^2=\frac{2k}{r+5k}\left(\frac{r+k}{2k}\right) \quad \textbf{[S41]}$$
$$\Rightarrow a=\sqrt{\frac{r+k}{r+5k}}\Rightarrow q=1-\frac{r+k}{r+5k}=\frac{4k}{r+5k}.$$

For an initial value $q(0)=q_0$, the optimal policy, $\boldsymbol{a}^*(\boldsymbol{q})$ will start at the corresponding point on the stable manifold of the interior equilibrium. This can be determined by explicitly using the formula for the cdf, F: In terms of the adjoint variable, ψ, we first calculate $G(u)=F^{-1}(u)=\sqrt{u}, g(u)=G'(u)=\frac{1}{2\sqrt{u}}$, so that $g^{-1}(y)=\frac{1}{4y^2}$. Along optimal trajectories,

$$u=g^{-1}\left(k\psi/q^2\right)=\frac{1}{4(k\psi/q^2)^2}=\frac{q^4}{4k^2\psi^2}\Rightarrow G(u)=\frac{q^2}{2k\psi}.$$

The equations of motion are, therefore,

$$\dot{q}=k\left(1-\frac{q^4}{4k^2\psi^2}-q\right), \quad \textbf{[S42]}$$

$$\dot{\psi}=(r+k)\psi-\frac{q^3}{k\psi}. \quad \textbf{[S43]}$$

The target equilibrium is given by the solution to

$$m\psi^2=\frac{q^3}{r+k}, \quad q=1-\frac{q^4}{4k^2\psi^2}=1-\frac{q^4(r+k)}{4kq^3}=1-\frac{(r+k)q}{4k},$$

which is consistent with the previous calculation. In the $q\psi$ plane, the target equilibrium is

$$q=\frac{4k}{r+5k}, \quad \psi=\sqrt{\frac{q^3}{k(r+k)}}=\frac{8k}{r+5k}\sqrt{\frac{1}{(r+5k)(r+k)}}.$$

See Fig. S3 for graphs of sample optimal trajectories when $k=1$, $r=0.5$.

Uniform Distribution of Consumer Types, Proof of Theorem 4

Consider an arbitrary target policy, π, and let s denote the corresponding target. Suppose that the initial state q is less than s. We first calculate the optimal target, s, starting from $q(0)=q$. Until it reaches s, $q(t)$ solves

$$q'(t)=k[1-q(t)]. \quad \textbf{[S44]}$$

The unique solution to the differential equation in **[S44]** with the initial condition $q(0)=q$ is

$$q(t)=1-e^{-kt}(1-q). \quad \textbf{[S45]}$$

As a consequence, if the initial state is $q<s$, the time T at which $q(T)=s$ solves

$$s=1-e^{-kT}(1-q), \quad \textbf{[S46]}$$

and therefore

$$T=\frac{1}{k}\ln\left(\frac{1-q}{1-s}\right). \quad \textbf{[S47]}$$

Therefore, under the policy π, the value function is

$$V_\pi(q)=sP(s)\left(\int_T^\infty e^{-rt}dt\right), \quad \textbf{[S48]}$$

where $P(s)=qA(s)$, which simplifies to

$$V_\pi(q)=\frac{1}{r}\left(\frac{1-q}{1-s}\right)^{-(r/k)}s^2(1-s). \quad \textbf{[S49]}$$

Eq. **S49** can be rewritten as

$$V_\pi(q)=\frac{1}{r}\left[(1-q)^{-(r/k)}\right]\left[s^2(1-s)^{\left(1+\frac{r}{k}\right)}\right]. \quad \textbf{[S50]}$$

Differentiating **[S50]** with respect to s yields

$$\frac{dV_\pi(q)}{ds}=\frac{1}{r}\left[(1-q)^{-(r/k)}\right]\left[2s(1-s)^{\left(1+\frac{r}{k}\right)}-\left(\left(1+\frac{r}{k}\right)s^2(1-s)^{\left(\frac{r}{k}\right)}\right)\right]. \quad \textbf{[S51]}$$

For $0<s<1$, the right-hand side of **[S50]** is strictly quasi-concave in s. Additionally, $\frac{dV_\pi(q)}{ds}=0$ at $s=0$ and $s=1$, which are minima for which $V_\pi(q)=0$ (in fact, both these statements are true for any function of the form $Kx^a(1-x)^b$ for $a, b\geq 1$).

As a consequence, the value $\sigma\in[0,1]$ that maximizes $V_\pi(q)$ with respect to s solves

$$2(1-\sigma)=\left(1+\frac{r}{k}\right)\sigma, \quad \textbf{[S52]}$$

which yields

$$\sigma=\frac{2k}{3k+r}. \quad \textbf{[S53]}$$

The corresponding price is

$$\frac{2k(k+r)}{(3k+r)^2}. \quad \textbf{[S54]}$$

(Note that when $r=0$, this yields the rational expectations equilibrium quantity $q^*=2/3$ and price $p^*=2/9$.)

Correspondingly, if the initial state is $q>s$, until it reaches s, $q(t)$ solves

$$q'(t)=-kq(t), \quad \textbf{[S55]}$$

which corresponds uniquely to the demand trajectory

$$q(t)=qe^{-kt}, \quad \textbf{[S56]}$$

and a similar computation yields the value function

$$V_\pi(q)=\frac{1}{2k+r}\left(q^2+q^{-\frac{r}{k}}s^{\left(2+\frac{r}{k}\right)}\left[\frac{2k(1-s)-rs}{r}\right]\right), \quad \textbf{[S57]}$$

which is also maximized with respect to s by the value of σ in **[S53]**.

Denote by π^* the target policy with target σ. We now show that the target policy π^* is optimal. For any given policy α (not necessarily a target policy), if its value function is continuously differentiable, then the corresponding Bellmanian functional is defined by

$$B_\alpha(q,a) = aq^2 - rV_\alpha(q) + V'_\alpha(q)m(q,a). \quad \textbf{[S58]}$$

According to a well-known proposition, a policy α is optimal if it satisfies the Hamilton–Jacobi–Bellman condition

$$B_\alpha(q,a) \le 0 \quad \text{for all} \quad q,a. \quad \textbf{[S59]}$$

(ref. 5, chap. 4). An alternative form for the last condition is

$$\alpha(q) = \arg\max_a B_\alpha(q,a). \quad \textbf{[S60]}$$

This follows from the fact that, for all q,

$$B_\alpha[q,\alpha(q)] = 0,$$

which is readily verified. (In fact, this identity is true for any stationary policy whose value function is C^1.)

Hence, from the above,

$$B_{\pi^*}\left[q,\pi^*(q)\right] = \pi^*(q)q^2 - rV_{\pi^*}(q) + k\left(1-q-\pi^*(q)\right)V'_{\pi^*}(q) = 0. \quad \textbf{[S61]}$$

It is useful to define

$$G(q) \equiv q^2 - kV'_\pi(q) \quad \textbf{[S62]}$$

and write $B_{\pi^*}(q,a)$ in the form

$$B_{\pi^*}(q,a) = aq^2 - rV_\pi(q) + k(1-q-a)V'_\pi(q) \quad \textbf{[S63]}$$

$$= -rV(q) + k(1-q)V'(q) + aG(q). \quad \textbf{[S64]}$$

Thus, $B_{\pi^*}(q,a)$ is linear in a, and the coefficient of a is $G(q)$. Hence

$$\arg\max_a B_{\pi^*}(q,a) = \begin{cases} 0, & \text{if} \quad G(q) < 0, \\ 1, & \text{if} \quad G(q) > 0. \end{cases} \quad \textbf{[S65]}$$

Recall that the stay-where-you-are policy is defined by

$$A(q) = 1-q. \quad \textbf{[S66]}$$

With this policy, $q(t) = q(0)$ for all $t > 0$, and its value function is

$$V_A(q) = \frac{\mathbb{A}(q)q^2}{r} = \frac{(1-q)q^2}{r}. \quad \textbf{[S67]}$$

Case 1: $0 < q < \sigma$. In this case $\pi^*(q) = 0$, and

$$B_{\pi^*}\left[q,\pi^*(q)\right] = -rV_{\pi^*}(q) + k(1-q)V'_{\pi^*}(q) = 0, \quad \textbf{[S68]}$$

so

$$kV'_{\pi^*}(q) = \frac{rV_{\pi^*}(q)}{1-q}, \quad \textbf{[S69a]}$$

$$G(q) = q^2 - \frac{rV_{\pi^*}(q)}{1-q}, \quad \textbf{[S69b]}$$

and from **[S67]**, it follows that

$$G(q) < 0 \Leftrightarrow V_{\pi^*}(q) > V_{A(q)}.. \quad \textbf{[S70]}$$

Suppose that the monopolist uses the policy π such that $a = 0$ for $0 \le t < u$ and then switches to the "stay-where-you-are" policy A from then on. Because her price is zero for $0 \le t \le u$, her resulting profit will be

$$g(u) \equiv e^{-ru}V_A[q(u)] = \left(\frac{1}{r}\right)e^{-ru}[q(u)]^2[1-q(u)], \quad \textbf{[S71]}$$

where $q(t)$ is determined by the differential equation $q'(t) = 1-q(t)$ on the interval [0, T), with $q(0) = q$. Note that

$$g(0) = V_A(q), \quad \textbf{[S72]}$$

$$g(T) = V_{\pi^*}(q), \quad \textbf{[S73]}$$

where, as before, T is the time at which $q(t)$ reaches the target σ under the policy π^*. Differentiating **[S71]** with respect to u and simplifying the resulting expression yields the derivative of g,

$$g'(u) = \left(\frac{1}{r}\right)e^{-ru}[q(u)][1-q(u)][2k-(3k+r)q(u)] > 0 \quad \textbf{[S74]}$$

for $0 \le u < T$, because

$$q(u) < \sigma = \frac{2k}{3k+r} \quad \text{for} \quad 0 \le u < T. \quad \textbf{[S75]}$$

Hence, $g(u)$ is strictly increasing in u and so

$$V_{\pi^*}(q) = g(T) > g(0) = V_A(q), \quad \textbf{[S76]}$$

and $B_{\pi^*}(q,a)$ is maximized at $a = 0$.

Case 2: $q > \sigma$. In this case $\pi^*(q) = 1$. Using an analogous argument, we find that

$$kV'_{\pi^*}(q) = \frac{-rV_{\pi^*}(q) + q^2}{q}, \quad \textbf{[S77]}$$

which leads to

$$G(q) > 0 \Leftrightarrow V_{\pi^*}(q) > V_A(q). \quad \textbf{[S78]}$$

The analogous expression for g is

$$g(u) \equiv q\int_0^u e^{-rt}q(t)dt + e^{-ru}V_A[q(u)] \quad \textbf{[S79]}$$

$$= q\int_0^u e^{-rt}q(t)dt + \left(\frac{1}{r}\right)e^{-ru}[q(u)]^2[1-q(u)], \quad \textbf{[S80]}$$

where $q(t)$ is defined by the differential equation

$$q'(t) = -kq(t), \quad q(0) = q \quad \textbf{[S81]}$$

in [0, T). Differentiating **[S80]** with respect to u yields

$$g'(u) = e^{-ru}\left([q(u)]^2 - [q(u)]^2[1-q(u)] + \frac{1}{r}\left(2q(u) - 3[q(u)]^2\right)q'(u)\right),$$

which simplifies to

$$g'(u) = \frac{q(u)}{r} e^{-ru}[(3k + r)q(u) - 2k]q(u), \quad \textbf{[S82]}$$

which is strictly positive, because

$$q(u) > \sigma = \frac{2k}{3k + r} \quad \text{for} \quad 0 \le u < T. \quad \textbf{[S83]}$$

Therefore, $g(u)$ is strictly increasing in u and so

$$V_{\pi^*}(q) = g(T) > g(0) = V_{\mathbf{A}}(q), \quad \textbf{[S84]}$$

and therefore $B_{\pi^*}(q, a)$ is maximized at $a = 1$.

Finally, note that, from **[S69a]** and **[S77]**,

$$V'_{\pi^*}(\sigma^-) = V'_{\pi^*}(\sigma^+) = V'_{\pi^*}(\sigma) = \frac{\sigma^2}{k}, \quad \textbf{[S85]}$$

$$G(\sigma) = 0, \quad \textbf{[S86]}$$

so V_{π^*} is continuously differentiable for all q. Hence B_{π^*} satisfies the Bellman optimality condition, which completes the proof.

From the derivation of the optimal target policy above, we get the value function of the optimal policy:

$$V_{\pi^*}(q) = \frac{1}{r}\left[(1-q)^{-(r/k)}\right]\left[\sigma(1-\sigma)^{\left(1+\frac{r}{k}\right)}\right]. \quad \textbf{[S87]}$$

A straightforward calculation shows that this expression is increasing in k if $q < \sigma$.

Concave Distributions of Consumer Types, Proof of Theorem 5

Consider any generalized target policy with target s, and let $V_s(q)$ be the corresponding value function. For $q(t) = s$,

$$q'(t) = k[1 - s - (sF[0] + (1-s)F[1])] = 0, \quad \textbf{[S88]}$$

and the immediate return is $s^2(1-s)$. Therefore,

$$V_s(s) = \int_0^\infty e^{-rt} s^2(1-s)dt = \frac{s^2(1-s)}{r}. \quad \textbf{[S89]}$$

When $q < s$,

$$q'(t) = k[1 - q(t)], \quad \textbf{[S90]}$$

with initial condition $q(0) = q$, yielding a solution

$$q(t) = 1 - e^{-kt}(1-q). \quad \textbf{[S91]}$$

The time T to get to s is therefore

$$T = \frac{1}{k}\log\left(\frac{1-q}{1-s}\right), \quad \textbf{[S92]}$$

and the value function is

$$V_s(q) = s^2(1-s)\left(\int_T^\infty e^{-rt}dt\right) \quad \textbf{[S93]}$$

or

$$V_s(q) = \frac{1}{r}\left[(1-q)^{-(\rho)}\right]\left[s^2(1-s)^{(1+\rho)}\right], \quad q < s. \quad \textbf{[S94]}$$

Similarly, when $q > s$,

$$q'(t) = -kq(t),$$

and a similar sequence of steps yields the value function for $q > s$:

$$V_s(q) = \frac{1}{2k + r}\left(q^2 + q^{-\rho}s^{(2+\rho)}\left[\frac{2k(1-s) - rs}{r}\right]\right), \quad q > s. \quad \textbf{[S95]}$$

As in the case of a uniform distribution, the value of s that maximizes $V_s(q)$ for both $q < s$ and $q > s$ is

$$\sigma = \frac{2k}{3k + r}. \quad \textbf{[S96]}$$

After some elementary simplification, the corresponding value function $V_\sigma(q)$ is

$$V_\sigma(q) = \begin{cases} \frac{1}{r}\left[(1-q)^{-(\rho)}\right]\left[\sigma^2(1-\sigma)^{(1+\rho)}\right], & q < \sigma, \\ \frac{\sigma^2(1-\sigma)}{r}, & q = \sigma, \\ \frac{1}{2k + r}\left(q^2 + \frac{k}{r}q^{-\rho}\sigma^{(3+\rho)}\right), & q > \sigma, \end{cases} \quad \textbf{[S97]}$$

$$\text{where } \rho = r/k. \quad \textbf{[S98]}$$

It is easily verified that $V_\sigma(q)$ is continuous at $q = \sigma$, and

$$V'_\sigma(\sigma^-) = V'_\sigma(\sigma^+) = \frac{\sigma^2}{k}, \quad \textbf{[S99]}$$

which verifies that V_σ is continuously differentiable. One also verifies that $V'_\sigma(q) > 0$ for all q (see below).

The Bellmanian corresponding to $V_\sigma(q)$ is

$$t\overline{B}(q, \psi) = q^2\overline{a}(\psi) - rV_\sigma(q) + kV'_\sigma\overline{F}(\psi), \quad \textbf{[S100]}$$

where ψ denotes a generic measure in Ψ. Because F is strictly concave and $V'_\sigma(q) > 0$, for each q the Bellmanian is maximized in ψ by taking a $\psi \in \Phi(q)$. By a slight abuse of notation, denote a measure in $\Phi(q)$ by the corresponding probability $\phi(q)$. Thus, for a measure $\phi \in \Phi(q)$, the Bellmanian equals

$$\overline{B}(q, \phi) = G(q)\phi - rV(q) + kV'(q)(1-q), \quad \textbf{[S101]}$$

where

$$G(q) \equiv q^2 - kV'_\sigma(q). \quad \textbf{[S102]}$$

The Bellmanian is therefore linear in ϕ and consequently is maximized by $\phi(q) = 0$ for $G(q) < 0$ and by $\phi(q) = 1$ for $G(q) > 0$.

Case 1: $q < \sigma$. In this case, one verifies that (after some simplification)

$$V'_\sigma(q) = \frac{\sigma^2}{k}\left(\frac{1-\sigma}{1-q}\right)^{(1+\rho)}, \quad q < \sigma, \quad \textbf{[S103]}$$

and therefore

$$G(q) = \frac{q^2(1-q)^{(1+\rho)} - \sigma^2(1-\sigma)^{(1+\rho)}}{(1-q)^{(1+\rho)}} \quad \text{for} \quad q < \sigma. \quad \textbf{[S104]}$$

Because $q < \sigma$, it follows that $G(q) < 0$, and therefore, $\phi(q) = 0$.

Case 2: $q > \sigma$. In this case, one verifies that (again after some simplification)

$$V'_\sigma(q) = \frac{1}{2k+r}\left[2q - \left(\sigma^2\left(\frac{\sigma}{q}\right)^{(1+\rho)}\right)\right], \qquad q > \sigma, \qquad \textbf{[S105]}$$

and therefore

$$G(q) = q^2 - \frac{k}{2k+r}\left[2q - \left(\sigma^2\left(\frac{\sigma}{q}\right)^{(1+\rho)}\right)\right] \quad \text{for} \quad q > \sigma, \qquad \textbf{[S106]}$$

which simplifies to

$$G(q) = q^2\left(\left[1 - \left(\frac{\sigma}{q}\right)^{3+\rho}\right] - \left[\left(\frac{3+\rho}{2+\rho}\right)\left(\frac{\sigma}{q}\right)\left(1 - \left(\frac{\sigma}{q}\right)^{2+\rho}\right)\right]\right) \qquad \textbf{[S107]}$$

for $q > \sigma$. This is of the form

$$K\left[\left(1 - x^{(y+1)}\right) - \frac{y+1}{y}x(1-x^y)\right], \text{with } x = \frac{\sigma}{q}, y = 2+\rho. \qquad \textbf{[S108]}$$

The identity

$$\frac{1-x^{(y+1)}}{1-x^y} > \frac{(y+1)}{y}x \quad \text{for} \quad y > 0, \quad 0 < x < 1 \qquad \textbf{[S109]}$$

establishes that $G(q) > 0$ and therefore $\phi(q) = 1$.

Finally,

$$G(\sigma) = 0, \qquad \textbf{[S110]}$$

and therefore $\overline{B}(\sigma, \phi)$ is constant in ϕ. The generalized target policy with target σ therefore satisfies the Hamilton–Jacobi–Bellman condition, which completes the proof.

An Evolving Consumer Population, Proof of Theorem 6

The law of motion can be rewritten as

$$m(q,a) = k[1 - a - hq], \quad 0 < q \le 1, \quad 0 \le a \le 1, \qquad \textbf{[S111]}$$

where

$$h \equiv 1 + \frac{c}{k} \ge 1. \qquad \textbf{[S112]}$$

Now consider an arbitrary target policy with target s. Starting at $q < s$, until $q(t)$ reaches s, $q(t)$ satisfies the differential equation

$$q'(t) = k[1 - hq(t)], \quad q(0) = q, \qquad \textbf{[S113]}$$

whose solution is

$$q(t) = \frac{1}{h}\left[1 - e^{-kht}(1-hq)\right], \qquad \textbf{[S114]}$$

and proceeding as in the proof of *Theorem 1* yields the value function

$$V(q) = \frac{1}{r}s^2[1-hs]^{1+\rho}(1-hq)^{-}\rho, \quad q < s, \qquad \textbf{[S115]}$$

$$\text{where } \rho = \frac{r}{hk}. \qquad \textbf{[S116]}$$

The right-hand side is strictly quasi-concave for $0 < s \le 1$, and it is maximized at

$$\sigma = \frac{2k}{3hk + r}, \qquad \textbf{[S117]}$$

which is our candidate optimal target. Correspondingly, starting at $q > s$, the value function solves to being

$$V(q) = \frac{1}{2hk+r}\left(q^2 + q^{-\rho}s^{(2+\rho)}\left[\frac{2k(1-hs) - rs}{r}\right]\right), \quad q > s, \qquad \textbf{[S118]}$$

which is also maximized in s at σ.

Now, the Bellmanian functional for the target policy with target σ is

$$B(q,a) = aq^2 - rV(q) + m(q,a)V'(q), \qquad \textbf{[S119]}$$

which simplifies to

$$B(q, a = a\left[q^2 - kV'(q)\right] - \left[rV(q) + k - khV'(q)\right]. \qquad \textbf{[S120]}$$

Recalling the function $G(q)$,

$$G(q) = q^2 - kV'(q), \qquad \textbf{[S121]}$$

it follows again that

$$\arg\max_a B(q,a) = \begin{cases} 0, & \text{if} \quad G(q) < 0, \\ 1, & \text{if} \quad G(q) > 0. \end{cases} \qquad \textbf{[S122]}$$

Differentiation of the value function V yields, after some rearranging,

$$V'(q) = \frac{\sigma^2}{k}\left(\frac{1-h\sigma}{1-hq}\right)^{[1+\rho]}, \qquad \textbf{[S123]}$$

which in turn implies that

$$G(q) = q^2\left(1 - \frac{f_0(\sigma)}{f_0(q)}\right), \quad q < \sigma, \qquad \textbf{[S124]}$$

where

$$f_0(x) \equiv x^2[1-hx]^{[1+\rho]}. \qquad \textbf{[S125]}$$

However, the function $f_0(x)$ is maximized at $x = \sigma$, which implies that $f_0(q) < f_0(\sigma)$ for $q < \sigma$, and therefore $G(q) < 0$ for $q < \sigma$. A similar computation, which is omitted, verifies that $G(q) > 0$ for $q > \sigma$. Finally, one can verify that $V_1(\sigma-, \sigma) = V_1(\sigma+, \sigma)$, and this completes the proof of *Theorem 6*.

Myopic and "Stubborn" Consumers, Proof of Theorem 7

Consider an arbitrary target policy with target s. We first characterize the optimal target, starting from $q(0) = q$. For the purpose of this section, it is convenient to take the control variables to be price, $p(t)$, rather than $a(t)$.

We begin with *Case 1*, in which $q < s$. Until $q(t)$ reaches s, the price is zero, and $q(t)$ satisfies the differential equation

$$q'(t) = k[1 - q(t)]. \qquad \textbf{[S126]}$$

When $q(t)$ reaches the target, s, the price jumps to the "stay as you are" price,

$$P(s) = (1-q)(\gamma s + (1-\gamma)\omega).$$

As in the uniform case, the value function for the target policy with target s is

$$V(q,s)=\left(\frac{1-s}{1-q}\right)^{\rho}\frac{P(s)s}{r} \quad \textbf{[S127]}$$

$$=\frac{f(s)}{r(1-q)^{\rho}}, \quad \textbf{[S128]}$$

$$\text{where } \rho=\frac{r'}{k}$$

$$f(s)=(1-s)^{\rho+1}\left[\gamma s^2+(1-\gamma)\omega s\right]. \quad \textbf{[S129]}$$

Hence the target that maximizes $V(q,s)$ is the value of s that maximizes $f(s)$. One verifies that

$$f'(s)=(1-s)^{\rho}G(s), \quad \text{where} \quad \textbf{[S130]}$$

$$G(s)=-\gamma(\rho+3)s^2+[2\gamma-(\rho+2)(1-\gamma)\omega]s+(1-\gamma)\omega.$$

Note that $f'(s)$ and $G(s)$ have the same sign. Also, G is quadratic and concave, and $G(0)=(1-\gamma)\omega>0$. Hence f is maximized at the larger of the two roots of $G(s)=0$. Call this root $\sigma(\gamma,\omega)$; it is the optimal target. Note that it is independent of the starting state, q.

We now show that, for $q<\sigma(\gamma,\omega)$, the target policy with target $\sigma(\gamma,\omega)$ is optimal among all policies. For the purpose of this proof, we define

$$\hat{\sigma}=\sigma(\gamma,\omega),$$

$$\hat{V}(q)=V(q,\hat{\sigma}).$$

The Bellmanian functional for this policy is

$$B(q,p)=pq-r\hat{V}(q))+k\left[1-\frac{p}{\gamma q+(1-\gamma)\omega}-q\right]\hat{V}'(q). \quad \textbf{[S131]}$$

Differentiating with respect to p, we have

$$\frac{\partial B}{\partial p}=q-\frac{k\hat{V}'(q)}{\gamma q+(1-\gamma)\omega}. \quad \textbf{[S132]}$$

Because

$$\hat{V}'(q)=\frac{f(\hat{\sigma})}{k(1-q)^{\rho+1}},$$

it follows that

$$\frac{\partial B}{\partial p}=q-\frac{f(\hat{\sigma})}{(1-q)^{\rho+1}[\gamma q+(1-\gamma)\omega]}.$$

Hence $\frac{\partial B}{\partial p}<0$ if and only if

$$(1-q)^{\rho+1}\left[\gamma q^2+(1-\gamma)\omega q\right]<f(\hat{\sigma}), \quad \text{or}$$
$$f(q)<f(\hat{\sigma}),$$

which is true for $q<\hat{\sigma}$. This completes the proof of the optimality of the target policy with target $\hat{\sigma}$ in *Case 1*. The argument for *Case 2*, $q>\hat{\sigma}$, is analogous and is omitted. Finally, one can verify that $\frac{\partial V}{\partial q}(\hat{\sigma}-,\hat{\sigma})=\frac{\partial V}{\partial q}(\hat{\sigma}+,\hat{\sigma})$. This completes the proof of condition *i* of *Theorem 7*.

To prove condition *ii*, write $G(s)$ in the form

$$\begin{aligned} G(s,\gamma)&=\gamma g_b(s)+(1-\gamma)g_a(s), \quad \text{where}\\ g_a(s)&=-(\rho+3)s^2+2s,\\ g_b(s)&=-(\rho+2)\omega s+\omega. \end{aligned} \quad \textbf{[S133]}$$

Recall that $\hat{\sigma}=\sigma(\gamma,\omega)$ is the larger root of

$$G[s,\gamma]=0.$$

A standard "comparative statics" calculation yields

$$\frac{\partial\sigma}{\partial\gamma}=-\frac{g_b[\sigma(\gamma,\omega)]-g_a[\sigma(\gamma,\omega)]}{\gamma g'_b[\sigma(\gamma,\omega)]+(1-\gamma)g'_a[\sigma(\gamma,\omega)]}. \quad \textbf{[S134]}$$

Let σ_b be the positive root of $g_b(s)=0$ (the other root is 0), and let σ_a be the root of $g_a(s)=0$. Then

$$\sigma_b=\frac{2}{\rho+3}, \quad \sigma_a=\frac{1}{\rho+2}. \quad \textbf{[S135]}$$

Note that

$$\frac{\sigma_b}{2}<\sigma_a<\sigma_b. \quad \textbf{[S136]}$$

Note also that (i) $g_b(s)$ is decreasing and positive for $\frac{\sigma_b}{2}\le s<\sigma_b$, ($ii$) $g_a(s)$ is decreasing and is negative for $\sigma_a<s\le\sigma_b$, and (iii) $\sigma_a<\sigma(\gamma,\omega)<\sigma_b$ for $0<\gamma<1$ (Fig. S4). Hence, by **[S134]** and **[S136]**, $\frac{\partial\sigma}{\partial\gamma}>0$ for $0<\gamma<1$, which completes the proof of *Theorem 7*.

To prove condition *iii* of *Theorem 7*, first note that, independent of the value of ω, condition *ii* of *Theorem 7* establishes that for $0<\gamma<1$,

$$\frac{1}{2+\rho}<\sigma(\gamma,\omega)<\frac{2}{3+\rho}. \quad \textbf{[S137]}$$

Also, from the second line of **[S130]**, σ is defined by

$$-\gamma(\rho+3)[\sigma(\gamma,\omega)]^2+[2\gamma-(\rho+2)(1-\gamma)\omega]\sigma(\gamma,\omega)+(1-\gamma)\omega=0. \quad \textbf{[S138]}$$

Differentiating both sides of **[S138]** with respect to ω and rearranging yields

$$\frac{\partial\sigma}{\partial\omega}=-\left(\frac{(1-\gamma)[(2+\rho)\sigma(\gamma,\omega)-1)}{2\gamma[(3+\rho)\sigma(\gamma,\omega)-1]+\omega[1-\gamma](2+\rho)}\right). \quad \textbf{[S139]}$$

From **[S137]**, $(2+\rho)\sigma(\gamma,\omega)>1$, and thus both the numerator and the denominator of the expression in parentheses on the right-hand side of **[S139]** are strictly positive. This completes the proof of condition *iii* of Theorem 7.

Additional Bibliographic Notes

These additional bibliographic notes go beyond those in the main text of the article.

Cabral et al. (6) study the dynamic pricing of a durable network good in a two-stage model with rational consumers, where they illustrate how the presence of network effects may overturn Coasian dynamics and lead to first-period pricing that is lower than second-period pricing. Fudenberg and Tirole (7) model dynamic pricing by a monopolist who sells a network good to overlapping generations of consumers who live for two periods, although they assume perfect rationality on the part of their consumers. Related papers that study single-period monopoly price discrimination based on a model of rational demand expectations include those by Oren et al. (8) and by Sundararajan (9, 10).

Shared information systems often display network effects, and our model may thus inform the literature on the adoption of such systems. For example, Riggins et al. (11) model the two-stage adoption of an interorganizational system with positive and negative adoption externalities. Although their model uses the standard notion of fulfilled expectations, they do discuss a case with myopic adopters. They show that subsidies are often necessary to induce adoption in the first stage, a result qualitatively similar to ours. Seidmann and Wang (12) examine a related problem for the adoption of Electronic Data Interchange in a two-sided network of buyers and suppliers, incorporating not just positive network effects from higher adoption, but also negative (or "competitive") externalities imposed by a buyer (supplier) on other buyers (suppliers) by their adoption; a similar trade-off is modeled by Westland as well (13).

A review of bounded rationality in economics is beyond the scope of this paper. The bounded rationality of agents in our model leads to a demand adjustment process that is "viscous" and is similar in this regard to the model of Radner (14) and Radner and Richardson (15). These papers, however, model a good of constant value and do not study network effects. Models of boundedly rational forecasting in macroeconomics are discussed in ref. 16. More general treatments of bounded rationality in economics can be found in refs. 17 and 18 and the references cited therein.

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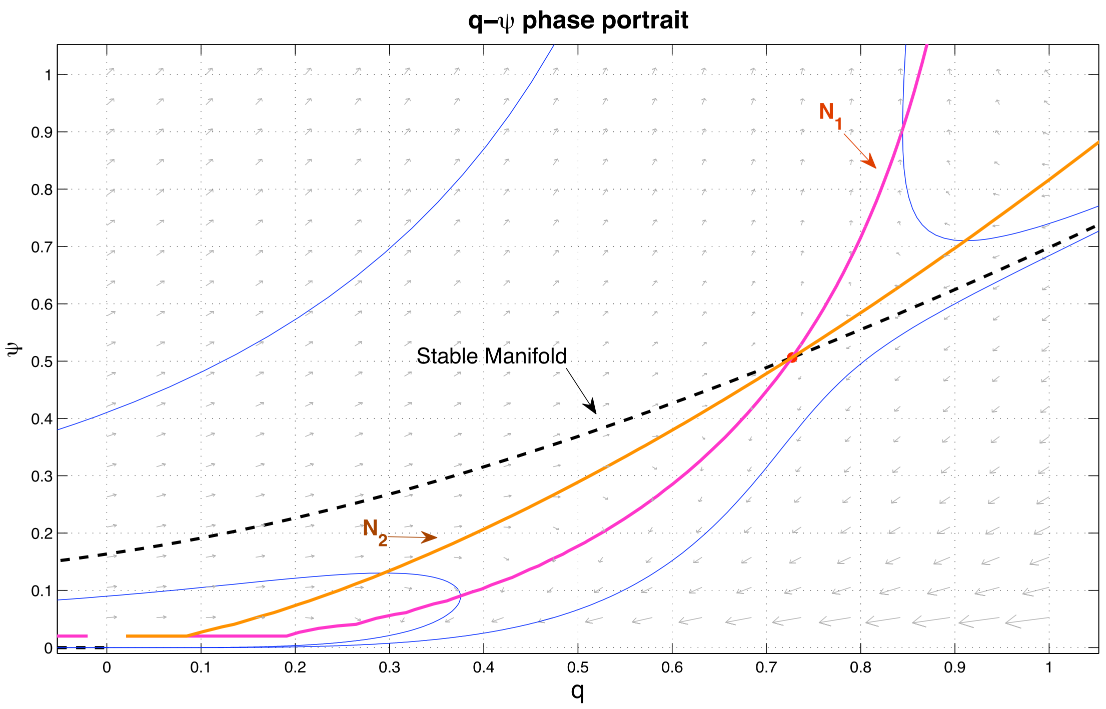

Fig. S1. Phase portrait for the system in the $q\psi$ plane with $F(a) = a^2$. There is a unique interior equilibrium, which is a saddle. All optimal trajectories lie on the stable manifold of this saddle, because other trajectories are either unbounded or remain at the equilibrium $q = \psi = 0$.

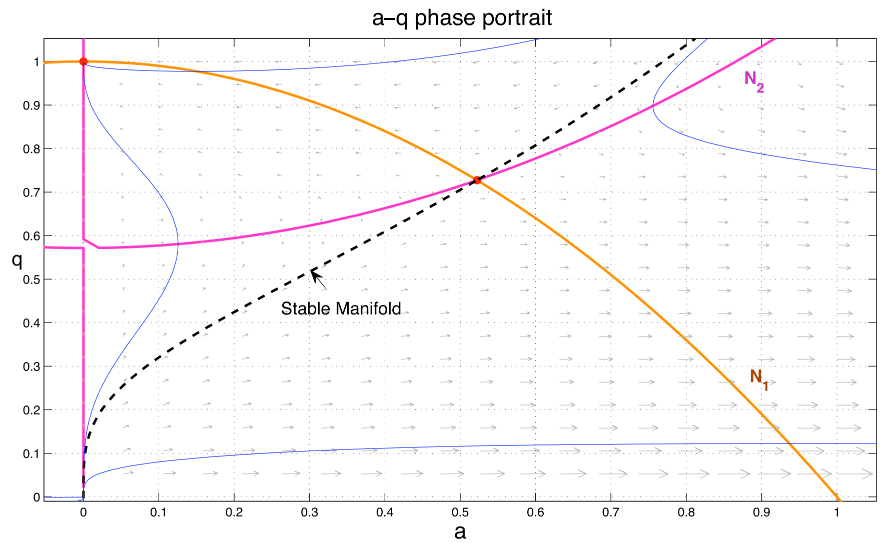

Fig. S2. Phase portrait in the aq plane showing the nullclines, $N_1 : \dot{q} = 0$ and $N_2 : \dot{a} = 0$. The interior equilibrium is a saddle, and the only optimal trajectories, where $0 < a < 1$, lie on the stable manifold of this equilibrium. In this example, $F(a) = a^2, k = 1, r = 0.5$.

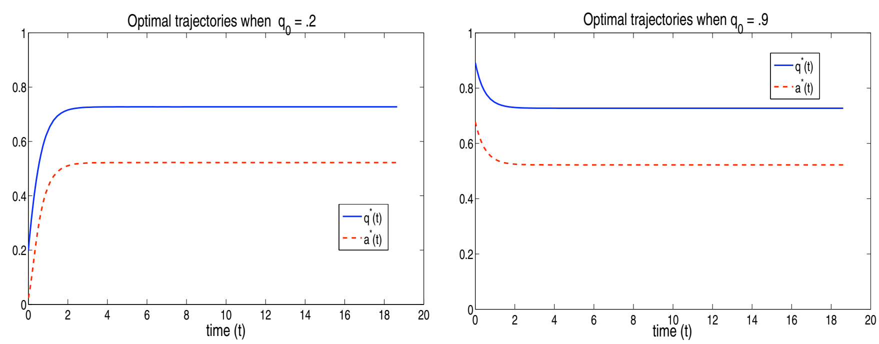

Fig. S3. Sample optimal trajectories in the quadratic case: $F(a) = a^2, k = 1, r = 0.5$. (*Left*) $q_0 = 0.2$; (*Right*) $q_0 = 0.9$.

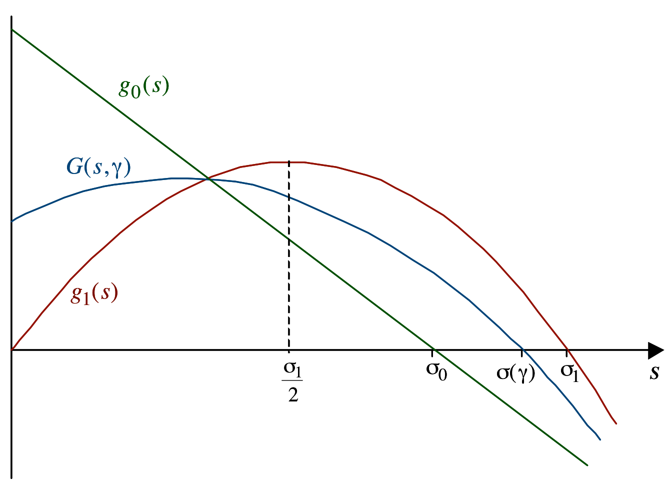

Fig. S4. Illustration of the proof of condition *ii* of *Theorem 7*.