SI Appendix

Technical supplement

Let L be the 10 by 20 matrix of data measurements from the smaller lattice on the retinal photo. In order to compare L with the predictions of the theory, the theoretical domain R with Cartesian coordinates x and y must be properly positioned over the retinal photo, by an orthogonal transformation and the parameters A and B must be specified. The Mathematica computational program is used to find the best least squares fit, using the following lines of code, followed by explanatory comments. The pseudocode for the larger data matrix K is similar.

Code for L:

TableForm[Table[0,{i,1,20},{j,1,10}]]

L=??

(here one inserts the observed data from the retinal lattice)

```
S =
 Table[
  180
  ─── ×
   π
   ArcTan[
    ((m × (-(.3 ) + j × (.2)) + u) /
        ((m × (-(.3 ) + j × (.2)) + u)^2 + (m × ((2.1) - i × (.2)) + v)^2) +
      A ((-(m × (-(.3 ) + j × (.2)) + u) + 1) /
          (((m × (-(.3 ) + j × (.2)) + u) - 1)^2 + (m × ((2.1) - i × (.2)) + v)^2)) + B) /
     (-(m × ((2.1) - i × (.2)) + v) /
        ((m × (-(.3 ) + j × (.2)) + u)^2 + (m × ((2.1) - i × (.2)) + v)^2) +
      A ((m × ((2.1) - i × (.2)) + v) /
          (((m × (-(.3 ) + j × (.2)) + u) - 1)^2 + (m × ((2.1) - i × (.2)) + v)^2)))] + 90,
  {i, 1, 20}, {j, 1, 10}]
```

```
P =
 Table[
  (If[L[[i, j]] == 0, 0, Min[Abs[S[[i, j]] - L[[i, j]]],
      180 - Abs[S[[i, j]] - L[[i, j]]]]])^2, {i, 1, 20}, {j, 1, 10}]
```

Minimize[{(Sum[P[[i,j]],{i,1,20},{j,1,10}]),0≤A≤.6,.7≤m≤1.1,-.1≤u≤.1,-.05≤v≤.05,0≤B≤.5},{A,B,m,u,v}]

(this will generate the A and B which minimize mean square deviation)

```
(              ??                                  )^(1/2)
( ──────────────────────────────────────────────── )
( Sum[(If[L[[i, j]] == 0, 0, 1]), {i, 1, 20}, {j, 1, 10}] )
```

(here one must insert the previously calculated sum of squares to calculate the standard deviation)

```
ContourPlot[
 ArcCos[x / √(x² + y²)] - A × ArcCos[(x - 1) / √((x - 1)² + y²)] + B × y /.
  {A →, B →}, {x, -.25, 1.75}, {y, 0, 2}, PlotPoints → 40,
 Contours → 30, ColorFunction → Hue]
```

(here enter A and B. This will generate a graphic of the theoretical curves)

In these calculations, S is a matrix of predicted values, whose entries are subtracted from corresponding L entries and then squared in the matrix P, with 0 entries whenever the L entry is zero (this explains why we standardized horizontal angles to 180). The entries of L are indexed by integers i and j, which are initially set for a lattice with minimal distances of .2. They are then related to x and y by the positioning parameters m, u and v.

x = mj + u
y = mi + v

The inversion between i and j occurs to match the particular labeling system for matrices in Mathematica (i indexes rows, which correspond to the y coordinate, and j indexes columns, corresponding to the x coordinate). A rotation by an angle θ was avoided by the earlier rotation of the retinal photo, although, in principle, it could be allowed to provide a small adjustment of photo positioning. Parameters m, u, and v specify the exact positioning of the morphogen sources.

The angles for α and β are obtained from x and y (or i and j) by using the inverse tangent function ArcTan. Some care must be taken to obtain the right branch of this multivalued function and to avoid singularities (zero in the denominator). Thus, the theoretically predicted angle is the value of

```
180 / π × ArcTan[∂_y (α - Aβ + By) / ∂_x (α - Aβ + By)] + 90
```

which lies in the interval (0,180].

The calculations for the streak potential are analogous. Note that the potential for a charge continuously distributed from –1 to +1 is

$$\frac{1}{2}\left(-4 + 2\,y\left(\mathrm{ArcTan}\left[\frac{1-x}{y}\right] + \mathrm{ArcTan}\left[\frac{1+x}{y}\right]\right) - (-1+x)\,\mathrm{Log}\left[(-1+x)^2 + y^2\right] + (1+x)\,\mathrm{Log}\left[(1+x)^2 + y^2\right]\right)$$

This function, scaled and translated to the interval from 1-s to 1+s will replace the macular point source. . Thus the potential is now the more complicated harmonic function

$$
\begin{aligned}
\Phi = {} & \mathrm{Log}\left[\sqrt{x^2+y^2}\right] - \\
& D \times \\
& \left(\frac{1}{s}\left(-\left(\frac{1}{s}\,y\right)\left(\mathrm{Log}\left[\sqrt{\left(\left(\frac{1}{s}\,x-\frac{1}{s}\right)-1\right)^2+\left(\frac{1}{s}\,y\right)^2}\right]\right)+\left(\left(\frac{1}{s}\,x-\frac{1}{s}\right)-1\right)\mathrm{ArcTan}\left[\frac{\left(\left(\frac{1}{s}\,x-\frac{1}{s}\right)-1\right)}{\left(\frac{1}{s}\,y\right)}\right]+\right. \\
& \left.\left(\frac{1}{s}\,y\right)\left(\mathrm{Log}\left[\sqrt{\left(\left(\frac{1}{s}\,x-\frac{1}{s}\right)+1\right)^2+\left(\frac{1}{s}\,y\right)^2}\right]\right)-\left(\left(\frac{1}{s}\,x-\frac{1}{s}\right)+1\right)\mathrm{ArcTan}\left[\frac{\left(\left(\frac{1}{s}\,x-\frac{1}{s}\right)+1\right)}{\left(\frac{1}{s}\,y\right)}\right]\right)\right)+C\times x
\end{aligned}
$$

The corresponding fibration equation becomes

$$
\begin{aligned}
& \mathrm{ArcTan}\left[\frac{y}{x}\right] - \\
& D\times\frac{1}{s}\left(-\left(\frac{1}{s}\,y\right)\left(\mathrm{Log}\left[\sqrt{\left(\left(\frac{1}{s}\,x-\frac{1}{s}\right)-1\right)^2+\left(\frac{1}{s}\,y\right)^2}\right]\right)+\left(\left(\frac{1}{s}\,x-\frac{1}{s}\right)-1\right)\mathrm{ArcTan}\left[\frac{\left(\left(\frac{1}{s}\,x-\frac{1}{s}\right)-1\right)}{\left(\frac{1}{s}\,y\right)}\right]+\right. \\
& \left.\left(\frac{1}{s}\,y\right)\left(\mathrm{Log}\left[\sqrt{\left(\left(\frac{1}{s}\,x-\frac{1}{s}\right)+1\right)^2+\left(\frac{1}{s}\,y\right)^2}\right]\right)-\left(\left(\frac{1}{s}\,x-\frac{1}{s}\right)+1\right)\mathrm{ArcTan}\left[\frac{\left(\left(\frac{1}{s}\,x-\frac{1}{s}\right)+1\right)}{\left(\frac{1}{s}\,y\right)}\right]\right)+C\times y = \\
& \mathrm{Constant}
\end{aligned}
$$