I. MODELING THE QD-QPC SYSTEM

To describe our experiment we use a model previously employed in Refs. [1–3]. We thus consider a master equation for the probability vector \( |p(n, t)\rangle = [p_{00}(n, t), p_{10}(n, t), p_{01}(n, t), p_{11}(n, t)]^T \), containing the probabilities \( p_{kl}(n, t) \) for the quantum dot (QD) to be in a charge state with \( k = 0, 1 \) extra electrons, and the quantum point contact (QPC) indicating that \( l = 0, 1 \) extra electrons reside on the QD, while \( n \) electrons have been counted during the time span \( t \). The probability distribution for the number of counted electrons is equal to the sum of the four probabilities. This can be written as the inner product of the probability vector with the vector \( |1\rangle = [1, 1, 1, 1] \), i.e., \( P(n, t) = \langle 1 | p(n, t) \rangle \). The corresponding moment generating function is defined as

\[
\hat{P}(z, t) = \sum_n P(n, t) e^{nz} = \langle 1 | \hat{p}(z, t) \rangle
\]

with \( \hat{p}(z, t) = \sum_n |p(n, t)\rangle e^{nz} \) and \( z \) being the counting field. The dynamics of \( \hat{p}(z, t) \) is governed by a master equation (see e.g. Ref. [4]) of the form

\[
d\hat{p}(z, t)/dt = M(z) \hat{p}(z, t)
\]

with solution

\[
\hat{p}(z, t) = e^{M(z) t} \hat{p}(z, 0)
\]

(1)

The unresolved (with respect to the number of passed electrons) probability distribution is \( \hat{P}(0, t) = \sum_n P(n, t) \), whose time evolution is determined by \( M(0) \). Using \( M(0) \) we can also calculate the waiting time distributions [5] for switches of the QPC current using the methods recently developed in Ref. [6]. For the probability of switching from a low to a high QPC current with a time difference \( \tau \) we find

\[
P_{\text{low}}(\tau) = \frac{\Gamma_Q \Gamma_D e^{-\left(\Gamma_S + \Gamma_D + \Gamma_Q\right)\tau/2}}{\sqrt{(\Gamma_S + \Gamma_D + \Gamma_Q)^2 - 4\Gamma_Q \Gamma_D}} \left( e^{\tau \sqrt{\left(\Gamma_S + \Gamma_D + \Gamma_Q\right)^2 - 4\Gamma_Q \Gamma_D}/2} - e^{-\tau \sqrt{\left(\Gamma_S + \Gamma_D + \Gamma_Q\right)^2 - 4\Gamma_Q \Gamma_D}/2} \right).
\]

(2)
For an ideal detector ($\Gamma_Q \to \infty$) the standard exponential law $P_{\text{low}}(\tau) = \Gamma_D e^{-\Gamma_D \tau}$ is recovered. The corresponding distribution for switches from a low to a high QPC current $P_{\text{high}}(\tau)$ is identical to $P_{\text{low}}(\tau)$ when $\Gamma_S$ and $\Gamma_D$ are interchanged. The theoretical predictions for the switching times are compared with experimental results, allowing us to extract the rates $\Gamma_S$, $\Gamma_D$, and $\Gamma_Q$ (see Fig. 1).

II. CUMULANT GENERATING FUNCTION

The cumulant generating function (CGF) is obtained as $S(z, \lambda) = \ln \hat{P}(z, t) = \ln \langle 1 | e^{M(z)t} | p(z, 0) \rangle$. The variable $\lambda$ denotes all system parameters including the time $t$. We assume that the system has reached the stationary state when counting begins, i.e., we take as the initial condition $|\hat{p}(z, 0)\rangle = |p_{\text{stat}}\rangle$, the (unique) normalized solution to $M(0)|p_{\text{stat}}\rangle = 0$. Whereas $\hat{P}(z, t)$ is an entire function, $S(z, \lambda)$ has logarithmic singularities at the zeros of $\hat{P}(z, t)$ in the complex-$z$ plane for any finite time $t$. However, a different non-analytic structure can still emerge in the long-time limit. For example, the CGF corresponding to an ideal detector ($\Gamma_Q \to \infty$) reads

$$S(z, \lambda) = \frac{\Gamma t}{1 + a} (w_{z,a} - 1) + \ln \frac{1 + q_{z,a} e^{-2w_{z,a}^2 \Gamma t}}{1 + q_{z,a}}$$

(3)

where $\Gamma \equiv \Gamma_S$, $w_{z,a} \equiv \sqrt{1 - a^2} e^z + a^2$, $q_{z,a} \equiv -\left(1 - w_{z,a} \right)^2$, and $a = (\Gamma_S - \Gamma_D) / (\Gamma_S + \Gamma_D)$ is the asymmetry parameter. The CGF can be decomposed in three terms $S(z, \lambda) = A(z) t + B(z) + C(z, t)$ in which $A(z)$ yields the limit of $S(z, \lambda)/t$ for $t \to \infty$ and $C(z, t)$ accounts for finite-time corrections, which are exponentially suppressed at long times. Clearly, $A(z)$ has branch points where the argument of the square-root is zero. However, at finite times, these branch point singularities are regularized by the presence of the finite-time correction $C(z, t)$, ensuring that only the logarithmic singularities of $C(z, t)$ remain [7]. In the long-time limit, the logarithmic singularities accumulate and produce the branch point singularities of $A(z)$. In this limit, the asymmetry parameter $a$ is the only relevant variable of the CGF. In contrast, at finite times, the analytic structure of the CGF is time-dependent in a non-trivial manner as discussed in the following section.

The cumulants of the distribution are determined by the derivatives of the CGF $S(z, \lambda)$ with respect to the counting field $z$ at the origin. In practice, it may be difficult to calculate
very high orders of these derivatives by direct differentiation, since the resulting expressions become very large. Instead, we determine the derivatives at \( z = 0 \) using a Cauchy integral, writing

\[
S(z, \lambda) = \frac{1}{2\pi i} \oint_C dz' \frac{S(z', \lambda)}{z'-z}.
\]

The cumulants are then

\[
\langle \langle n^m \rangle \rangle(\lambda) = S^{(m)}(0, \lambda) = \frac{m!}{2\pi} \int_{\pi}^{-\pi} d\theta e^{-im\theta} \frac{S(\varepsilon e^{i\theta}, \lambda)}{\varepsilon^m}.
\]

Deforming the contour into a circle with radius \( \varepsilon \), small enough that no singularities of \( S \) are enclosed, we may parametrize it as \( C : z = \varepsilon e^{i\theta}, \theta \in [-\pi, \pi] \). The cumulants can then be written as

\[
\langle \langle n^m \rangle \rangle(\lambda) = S^{(m)}(0, \lambda) = \frac{m!}{2\pi} \int_{-\pi}^{\pi} d\theta e^{-im\theta} \frac{S(\varepsilon e^{i\theta}, \lambda)}{\varepsilon^m}.
\]

While this approach suffices for the given problem, more sophisticated methods are also available for more complicated cases [8]. The model calculations shown in Fig. 2 were obtained by evaluating numerically the expression in Eq. (6) using the full expression for the CGF in the case of a finite detector bandwidth.

**III. TRANSIENT OSCILLATIONS**

For times that are not much larger than the relaxation time \( 1/(\Gamma_S + \Gamma_D) \), time enters as a relevant parameter in the CGF \( S(z, \lambda) \) and the corresponding singularities are in general time-dependent. This is the reason for the oscillatory dependence of the cumulants on time. In the long-time limit, the dependence of the singularities on the asymmetry parameter is relatively simple. At finite times, the dependence of the singularities on time and the asymmetry parameter is more complicated as we shall see.

Since \( S(z, \lambda) \) is periodic by construction, \( S(z + 2\pi i, \lambda) = S(z, \lambda) \), it is sufficient to restrict ourselves to the strip \( -\pi \leq \Im z \leq \pi \) containing the origin. The transient part of the CGF, denoted \( C(z, t) \) in the previous section, has time-dependent logarithmic singularities that coincide with the zeroes of the function \( Q(z, \lambda) = 1+q_{z,a} \exp(-\frac{2u}{1+a} \Gamma t) \), see Eq. (3). They all lie on the line \( \Im z = \pi \) and one finds that \( Q(x+\pi i, \lambda) > 0 \), whenever \( x < x_0 = \ln[a^2/(1-a^2)] \).

Therefore, all the singularities are localized rightwards from the dominating singularity of the non-transient part \( z_0 = x_0 + \pi i \) (see Fig. 4a). Considering now \( x > x_0 \), we get

\[
Q(x + \pi i, \lambda) = 1 - \exp i \left( 4 \arctan \frac{1}{u} - \frac{2u}{1+a} \Gamma t \right)
\]
with \( u = \sqrt{(1 - a^2) e^z - a^2} \), and thus find infinitely many (time-dependent) zeroes at \( z_{k,0}(t) = \ln[(a^2 + u_k^2)/(1 - a^2)] + \pi i \), by solving the equations

\[
\frac{2u_k}{1 + a} \Gamma t - 4 \arctan \frac{1}{u_k} = 2\pi(k - 1), \quad k = 1, 2, 3, \ldots
\]

The resulting picture now depends on whether the asymmetry parameter \( a \) is larger or smaller than the critical value \( a^* = 1/\sqrt{2} \) for which \( x_0 = 0 \). In the high-asymmetry regime, \( |a| > a^* \), the dominating time-dependent singularity is \( z_{1,0}(t) \), and its complex conjugate \( \overline{z_{1,0}(t)} = z_{1,-1}(t) \) (see Fig. 4a). The CGF can then be written \( S(z, \lambda) = S_{\text{reg}}(z, \lambda) + \ln[z - z_{1,0}(t)][z - z_{1,-1}(t)] \) with \( S_{\text{reg}}(z, \lambda) \) being non-singular around \( z_{1,0}(t) \) and \( z_{1,-1}(t) \). For a fixed finite time interval and in the large cumulant-order asymptotics, other time-dependent singularities become suppressed and the logarithmic singularities at \( z_{1,0}(t), z_{1,-1}(t) \) become responsible for the emergence of cumulant oscillations.

The low-asymmetry regime, \( |a| < 1/\sqrt{2} \), corresponding to the actual experimental setup, leads to a somewhat more involved analytical structure, since the logarithmic singularities now may have a negative real part. The dominating role is then played by the singularities on the line \( \Im z = \pi \) (and on the line \( \Im z = -\pi \)) as they move by the origin (see Fig. 4a) with time. For the experimentally relevant time span, the oscillations are governed by the first pair of singularities \( z_{1,0}(t), z_{1,-1}(t) \). For longer times, a rather complicated oscillation pattern is expected due to the interference with the next pairs of singularities that come close to the origin.

The long-time asymptotics of any fixed cumulant does not follow simply from the above finite-time analysis: as time increases the gap separating the dominating singularities from the others approaches zero and, in fact, the logarithmic singularities tend to form a continuous spectrum. Consequently, the smallest cumulant order for which our finite-time theory is meaningful also increases with time. However, in the limit of \( t \) going to infinity, we know the limit of \( S(z, \lambda)/t \) explicitly as it is given by \( A(z) \); its branch point singularities then determine the cumulant asymptotics. Apparently, we consider here two different asymptotic regimes, namely, the transient and the stationary ones, which correspond to the limits of large cumulant order and of large time taken in the opposite order: for the former, the large order asymptotics of the cumulants is considered first, whereas, for the latter, the first limit is that of large time. The different structure of the finite-time and of the stationary singularities then explains why the two limits are not interchangeable. Naturally, in interpreting
the time dependence of a moderate-order cumulant, one has to take into account that it exhibits a non-trivial crossover between both asymptotic regimes.


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