latter, however, preserving its proportion to the body as a whole, attains ultimately a much greater absolute size.

The fore limb bud of A. punctatum, engrafted in normal position on a tigrinum embryo, develops at first much more rapidly than the natural limb on the opposite side of the host, but is retarded in comparison with the limb of the donnor left in place. In the reciprocal transplantation the tigrinum limb bud remains for a time small, but its growth is soon greatly accelerated and the limb later becomes much larger than that of either species in its normal surroundings.

These results may be explained by the assumption that two factors are concerned in the growth process: the growth potential, a property of the cells of the graft; and a regulator, probably an internal secretion of the host, carried to the limb through the circulation.

[1] Cope, E. D. 1889. "The Batrachia of North America." *Bull. U. S. Nat. Mus.*, No. 34.

[2] Harrison, Ross G. 1918. "Experiments on the Development of the Forelimb of Ambylstoma, a Self Differentiating Equipotential System." *J. Exp. Zoöl.*, **25**.

[3] Detwiler, S. R. 1920a. "On the Hyperplasia of Nerve Centers Resulting from Excessive Peripheral Loading." *Proc. Nat. Acad. Sci.*, **6**, pp. 96–101. Also 1920b. "Functional Regulation in Animals with Composite Spinal Cords." *Ibid.*, pp. 695–700.

SETS OF COMPLETELY INDEPENDENT POSTULATES FOR CYCLIC ORDER[1]

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The "universe of discourse" in this paper consists of all systems (K,R), where K is a class of elements, $A,B,C,\ldots$, and R is a triadic relation, denoted by R(ABC), or simply ABC.

Within this universe of discourse, two classes of systems (K,R) are of special importance; first, the *betweenness-systems* characterized by any one of twelve sets of postulates, studied elsewhere;[2] and second, the *cyclic-order-systems* characterized by any one of three sets of postulates studied in the present paper.

The three sets of postulates for cyclic order are:

(1) E, B, C, D, 2,
(2) E, B, C, D, 3,
(3) E, B, C, D, 9;

each set consisting of five postulates selected from a certain basic list (see below).

The most familiar example of a cyclic-order-system—that is, a system (K,R) in which the class K is cyclically ordered by the relation R—is the system in which K is a class of points on the circumference of a circle, and R(ABC) means that the points A, B, C, are distinct and that the arc A-B-C, read clockwise, is less than 360°.

Basic List of Postulates for Cyclic Order.—The numbering of these postulates is made to conform with the numbering already used in the papers on betweenness. Postulates E, B, C, D concern three elements; postulates 2, 3, 9 concern four elements.

POSTULATE E. *If A, B, C, are distinct, then ABC implies BCA.*

POSTULATE B. *If A,B,C, are distinct, then at least one of the orders ABC, BCA, CAB, CBA, ACB, BAC, is true.*

POSTULATE C. *If A,B,C, are distinct, then ABC and ACB cannot both be true.*

POSTULATE D. *If ABC is true, then A,B, and C are distinct.*

POSTULATE 2. *If A,B,X,Y, are distinct, and XAB and AYB, then XAY.*

POSTULATE 3. *If A,B,X,Y are distinct, and XAB and AYB, then XYB.*

POSTULATE 9. *If A,B,C,X are distinct, and ABC is true, then at least one of the orders ABX and XBC will be true.*

This basic list is essentially exhaustive as far as "general laws" are concerned ("existence postulates" not being considered); but it contains a number of redundancies, as indicated in the following *theorems on deducibility:*

THEOREM 2_1. *Postulate 2 follows from E,C,9.*

To prove: If XAB and AYB, then XAY.

By E, AYB gives BAY. By 9, XAB with Y gives either XAY or YAB. But if YAB, then, by E, BYA, which conflicts with BAY, by C. Hence XAY.

THEOREM 2_2. *Postulate 2 follows from E, 3.*

To prove: If XAB and AYB, then XAY.

By E, XAB gives BXA; and by E, AYB gives YBA. Then by 3, YBA and BXA give YXA, whence, by E, XAY.

THEOREM 3_1. *Postulate 3 follows from E,C,9.*

To prove: If XAB and AYB, then XYB.

By E, XAB gives ABX. By 9, ABX with Y gives either ABY or YBX. But ABY conflicts with AYB, by C. Therefore YBX, whence, by E, XYB.

THEOREM 3_2. *Postulate 3 follows from E, 2.*

To prove: If XAB and AYB, then XYB.

By E, XAB gives BXA; and by E, AYB gives YBA. Then by 2, YBA and BXA give YBX, whence, by E, XYB.

THEOREM 9_1. *Postulate 9 follows from E,B,2.*

To prove: If *ABC*, then either *ABX* or *XBC*.

By E, *ABC* gives *BCA*. By B and E, either *ABX* or *XBA* must be true. But by 2, *XBA* with *BCA* leads to *XBC*. Hence either *ABX* or *XBC*.

THEOREM 9_2. *Postulate 9 follows from E,B,3.*

To prove: If *ABC*, then either *ABX* or *XBC*.

By E, *ABC* gives *CAB*. By B and E, either *ABX* or *AXB* must be true. But by 3, *CAB* with *AXB* leads to *CXB*, and hence, by E, to *XBC*. Therefore either *ABX* or *XBC*.

These theorems show that the three sets of postulates are equivalent.

Examples for Proving Independence.—We now exhibit examples of systems (K,R) having the properties described in the following tables, in which a plus sign (+) indicates that a postulate is satisfied, and a minus sign (−) that it is not satisfied.

Example 0 shows that all the postulates are *consistent.*

Examples 1, 2, 4, 16, 8 show that the five postulates of each set are independent in the ordinary sense, and examples 0–31 show that the five postulates of each set are *completely* independent in the sense of E. H. Moore.

The remaining examples enable us to see that among the seven postulates of our basic list there are no theorems on deducibility other than those proved above. Thus:

Postulate 2 is not deducible from E,B,C,D (Ex. 8); nor from E,B,D,9 (Ex. 36); nor from B,C,D,3,9 (Ex. 32).

Postulate 3 is not deducible from E,B,C,D (Ex. 8); nor from E,B,D,9 (Ex. 36); nor from B,C,D,2,9 (Ex. 33).

Postulate 9 is not deducible from E,B,C,D (Ex. 8); nor from E,C,D,2,3 (Ex. 35); nor from B,C,D,2,3 (Ex. 34).

Also, no one of the postulates E,B,C,D, is deducible from the remaining six (Exs. 1, 2, 4, 16).

Two General Theorems on Cyclic Order.—The following general theorems on cyclic order are of interest.

THEOREM I. *If A, B, C are distinct elements, in the order ABC, and if X is any other element, distinct from A, B, and C, then one and only one of the orders AXB, BXC, CXA, will be true.* (From E, C, 9).

In brief any cyclically ordered class is divided by any three of its elements into three non-overlapping subclasses.

Proof. In the first place, at least one of the orders *AXB*, *BXC*, *CXA* must be true. For, by 9, *ABC* with *X* gives either *ABX* or *XBC*, whence, by E, either *BXA* or *CXB*. But by 9, *BXA* with *C* gives *BXC* or *CXA;* and by 9, *CXB* with *A* gives *CXA* or *AXB*. Hence in any case, either *AXB* or *BXC* or *CXA* will be true.

The requisite examples are the following, the class K consisting in each case of four elements, 1, 2, 3, 4, and the relation R being defined by listing explicitly the true triads in each case.

Ex. 0. 123, 231, 312; 124, 241, 412; 134, 341, 413; 234, 342, 423.

Ex. 1. 123, 124, 134, 234.

Ex. 2. No triads true.

Ex. 3. 123, 124.

Ex. 4. All twenty-four permutations true.

Ex. 5. 123, 124, 134, 234; 132, 142, 143, 243.

Ex. 6a. 123, 231, 312; 321, 132, 213; 124, 241, 412; 421, 214, 142; 234, 342, 423; 432, 324, 243.

Ex. 6b. 123, 231, 312; 321, 132, 213.

Ex. 7. 123, 132, 124, 134.

Ex. 8. 123, 231, 312; 214, 142, 421; 134, 341, 413; 432, 324, 243.

Ex. 9. 123, 214, 134, 243.

Ex. 10. 123, 231, 312; 214, 142, 421.

Ex. 11. 123, 243.

Ex. 12. 123, 231, 312; 214, 142, 421; 134, 341, 413; 432, 324, 243; 321, 132, 213.

Ex. 13. 213, 231; 124, 142; 431, 413; 243, 234.

Ex. 14. 123, 231, 312; 321, 213, 132; 214, 142, 421.

Ex. 15. 123, 132; 413.

Exs. 16–31. Same as Exs. 0–15, with the triad 444 added.

Ex. 32. 123, 142, 143, 243; 423.

Ex. 33. 123, 341, 243, 124; 241.

Ex. 34. 123, 142, 234, 134.

Ex. 35. 123, 231, 312.

Ex. 36. 123, 231, 312; 321, 213, 132; 421, 214, 142; 431, 314, 143; 234, 342, 423; 432, 324, 243.

Ex.	E	B	C	D	2 3 9
0	+	+	+	+	+
1	−	+	+	+	+
2	+	−	+	+	+
3	−	−	+	+	+
4	+	+	−	+	+
5	−	+	−	+	+
6*	+	−	−	+	+
7	−	−	−	+	+
8	+	+	+	+	−
9	−	+	+	+	−
10	+	−	+	+	−
11	−	−	+	+	−
12	+	+	−	+	−
13	−	+	−	+	−
14	+	−	−	+	−
15	−	−	−	+	−
16	+	+	+	−	+
17	−	+	+	−	+
18	+	−	+	−	+
19	−	−	+	−	+
20	+	+	−	−	+
21	−	+	−	−	+
22*	+	−	−	−	+
23	−	−	−	−	+
24	+	+	+	−	−
25	−	+	+	−	−
26	+	−	+	−	−
27	−	−	+	−	−
28	+	+	−	−	−
29	−	+	−	−	−
30	+	−	−	−	−
31	−	−	−	−	−

*Examples 6a and 22a satisfy postulate 9; Examples 6b and 22b satisfy postulates 2 and 3.

Ex.	E B C	D	2 3 9
32	−++	+	−++
33	−++	+	+−+
34	−++	+	++−
35	+−+	+	++−
36	++−	+	−−+

In the second place, if any one of these three relations is true, both the other two will be false. For example, let AXB be true. Then by E, XBA and BAX. Now by 9, XBA with C gives either XBC or CBA, whence, by E, either BCX or ACB; but ACB conflicts with ABC, by C; hence BCX. Again, by 9, BAX with C gives either BAC or CAX; but BAC leads, by E, to ACB, which conflicts with ABC, by C; hence CAX. But if BCX and CAX are true, then by C, BXC and CXA must be false.

THEOREM II. *If A, B, C are distinct elements, in the order ABC; and if X, Y, Z are three other distinct elements such that AXB, BYC, CZA; then XYZ.* (From E, C, 9.)

Proof. By 9, ABC with Y gives either ABY or YBC; but YBC leads, by E, to BCY, which conflicts with BYC, by C; therefore ABY; whence by E, YAB. Then by 9, YAB with X gives either YAX or XAB; but XAB leads, by E, to ABX, which conflicts with AXB, by C; therefore YAX; whence by E, AXY.

By E, ABC gives BCA, and by 9, BCA with Y gives either BCY or YCA; but BCY conflicts with BYC, by C; therefore YCA; whence by E, CAY. Then by 9, CAY with Z gives either CAZ or ZAY; but CAZ conflicts with CZA, by C; therefore ZAY; whence by E, AYZ. Then by 9, AYZ with X gives either AYX or XYZ.

But AYX conflicts with AXY, by C. Hence XYZ.

The Relation Between Cyclic Order and Betweenness.—If we interchange two letters in Postulate E, so that it becomes

POSTULATE A. *If A, B, C are distinct, then ABC implies CBA;* then the five postulates A, B, C, D, 9 form a set of completely independent postulates for *betweenness* (*loc. cit.*, set 12).

[1] Presented to the American Mathematical Society on the following dates: the first set, December 28, 1916 [for brief abstract, see E. V. Huntington, "A Set of Independent Postulates for Cyclic Order," These PROCEEDINGS, **2**, pp. 630–631, 1916]; the second set (with proofs of complete independence for the first two sets), October 27, 1923; the third set, December 28, 1923.

[2] For the first eleven sets of postulates for betweenness, see E. V. Huntington and J. R. Kline, "Sets of Independent Postulates for Betweenness," *Trans. Amer. Math. Soc.*, **18**, pp. 301–325, 1917. For the twelfth set, involving the new postulate 9, see a forthcoming paper by E. V. Huntington, "A New Set of Postulates for Betweenness, with Proof of Complete Independence," presented to the Society, December 28, 1923, and to appear in *Trans. Amer. Math. Soc.*