Consistency of a counterexample to Naimark's problem

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We construct a C*-algebra that has only one irreducible representation up to unitary equivalence but is not isomorphic to the algebra of compact operators on any Hilbert space. This answers an old question of Naimark. Our construction uses a combinatorial statement called the diamond principle, which is known to be consistent with but not provable from the standard axioms of set theory (assuming that these axioms are consistent). We prove that the statement "there exists a counterexample to Naimark's problem which is generated by $\aleph_1$ elements" is undecidable in standard set theory.

Let $\mathcal{K}(H)$ denote the C*-algebra of compact operators on a complex, not necessarily separable, Hilbert space H. In ref. 1 Naimark observed that every irreducible representation (irrep) of $\mathcal{K}(H)$ is unitarily equivalent to the identity representation, so that each of these algebras has only one irrep up to equivalence, and in ref. 2 he asked whether this property characterizes the algebras $\mathcal{K}(H)$. In other words: if $\mathcal{A}$ is a C*-algebra with only one irrep up to unitary equivalence, is $\mathcal{A}$ isomorphic to some $\mathcal{K}(H)$? We call any algebra $\mathcal{A}$ that satisfies the premise of this question but not its conclusion a *counterexample to Naimark's problem.*

The problem was quickly settled in the separable case. Building on results in ref. 2, Rosenberg (3) showed that there are no separable (indeed, no separably acting) counterexamples to Naimark's problem. Around the same time, Kaplansky (4) introduced the so-called "type I" (or "GCR") C*-algebras and began developing their representation theory. (See refs. 5–7 for general background on type I C*-algebras.) This development was carried further by Fell and Dixmier, who showed in particular that any two irreps of a type I C*-algebra with the same kernel are unitarily equivalent (8) and conversely, any separable C*-algebra that is not type I has inequivalent irreps with the same kernel (9). As it is easy to see that no type I C*-algebra can be a counterexample to Naimark's problem, the latter result (partially) recovers Rosenberg's theorem, doing so, moreover, in the context of a general theory.

Next, in a celebrated paper Glimm (10) gave several characterizations of separable type I C*-algebras, showing in particular that every separable C*-algebra which is not type I has *uncountably many* inequivalent irreps. Since counterexamples to Naimark's problem cannot be type I, this reestablished the nonexistence of separable counterexamples in an especially dramatic way.

Some of Glimm's results were initially proven without assuming separability, and subsequent work by Sakai (11, 12) went further in this direction. However, separability was never removed from the reverse implication of the equivalence "type I $\Leftrightarrow$ irreps with the same kernel are equivalent," and it was recognized that Naimark's problem potentially represented a fundamental obstruction to a nonseparable generalization of this result. However, it was reasonable to expect that there were no counterexamples to Naimark's problem because it seemed likely that the separability assumption could be dropped in Glimm's theorem that non-type I C*-algebras have uncountably many inequivalent irreps, but this was never achieved.

That is the background for the present investigation. We construct a (necessarily nonseparable and not type I) counterexample to Naimark's problem. Our construction is not carried out in ZFC (Zermelo–Frankel set theory with the axion of choice): it requires Jensen's "diamond" principle, which follows from Gödel's axiom of constructibility and hence is relatively consistent with ZFC, i.e., if ZFC is consistent then so is ZFC plus diamond (see refs. 13 and 14 for general background on set theory). The diamond principle has been used to prove a variety of consistency results in mainstream mathematics (see, e.g., refs. 15–17).

Presumably, the existence of a counterexample to Naimark's problem is independent of ZFC, but we have not yet been able to show this. We can prove the relative consistency of the assertion "no C*-algebra generated by $\aleph_1$ elements is a counterexample to Naimark's problem"; indeed, this follows easily from ref. 10. Since our counterexample is generated by $\aleph_1$ elements, it follows that the existence of an $\aleph_1$-generated counterexample is independent of ZFC (see Corollary 7).

The basic idea of our construction is to create a nested transfinite sequence of separable C*-algebras $\mathcal{A}_\alpha$, for $\alpha < \aleph_1$, each equipped with a distinguished pure state f_α, such that for any $\alpha < \beta$, f_β is the unique extension of f_α to a state on $\mathcal{A}_\beta$. At the same time, for each α (or at least "enough" α) we want to select a pure state g_α on $\mathcal{A}_\alpha$ that is not equivalent to f_α, such that g_α has a unique state extension g'_α to $\mathcal{A}_{\alpha+1}$, and g'_α is equivalent to $f_{\alpha+1}$. Thus, we build up a continually expanding pool of equivalent pure states in an attempt to ensure that all pure states on $\mathcal{A} = \cup\mathcal{A}_\alpha$ will be equivalent.

There are two significant difficulties with this approach, which incidentally is essentially the only way an $\aleph_1$-generated counterexample to Naimark's problem could be constructed. First, there is the technical problem of finding a suitable algebra $\mathcal{A}_{\alpha+1}$ that contains $\mathcal{A}_\alpha$ and such that f_α and g_α extend uniquely to $\mathcal{A}_{\alpha+1}$ and become equivalent. We accomplish this via a crossed product construction which takes advantage of a powerful recent result of Kishimoto, Ozawa, and Sakai that, together with earlier work of Futamura, Kataoka, and Kishimoto, ensures the existence of automorphisms on separable C*-algebras that relate inequivalent pure states in a certain manner. The second fundamental challenge is to choose the states g_α in such a way that *all* pure states are eventually made equivalent to one another. This is especially troublesome because pure states can be expected to proliferate exponentially as α increases (every pure state on $\mathcal{A}_\alpha$ extends to at least one, but probably many, pure states on $\mathcal{A}_{\alpha+1}$) and we can only make a single pair of states equivalent at each step. We handle this issue by using the diamond principle, one version of which states that it is possible to select a single vertex from each level of the standard tree of height and width $\aleph_1$, such that every path down the tree contains in some sense "many" selected vertices. In our application the vertices of the tree at level α model the states on $\mathcal{A}_\alpha$ and diamond informs us how to choose g_α. Then every pure state

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Abbreviations: irrep, irreducible representation; ZFC, Zermelo–Frankel set theory with the axiom of choice.

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on $\mathcal{A}$ induces a path down the model tree, and hence its restriction to some $\mathcal{A}_\alpha$ equals g_α, so that all pure states are indeed taken care of at some point in our construction.

Unique Extension of Pure States to Crossed Products

In this section we consider the problem: if f and g are inequivalent pure states on a C*-algebra $\mathcal{A}$, is it possible to find a C*-algebra $\mathcal{B}$ that contains $\mathcal{A}$ such that (*i*) f and g have unique state extensions f' and g' to $\mathcal{B}$ and (*ii*) f' and g' are equivalent? We find that the answer is yes if A is simple, separable, and unital, and moreover we can ensure that $\mathcal{B}$ is also simple, separable, and unital, with the same unit as $\mathcal{A}$.

A net (a_λ) of positive, norm-one elements of a C*-algebra $\mathcal{A}$ is said to *excise* a state f on $\mathcal{A}$ if $\|a_\lambda x a_\lambda - f(x)a_\lambda^2\| \to$ for 0 all $x \in \mathcal{A}$ (ref. 18, p. 1239). By ref. 18, proposition 2.2, for each pure state f of a C*-algebra $\mathcal{A}$, there is a decreasing net (a_λ) that excises f. As described in ref. 18, p. 1240 (see also ref. 6, p. 87), the support projection p of f is a rank 1 projection in $\mathcal{A}^{**}$. Further, as shown in the last 7 lines of the proof of ref. 18, proposition 2.3, $a_\lambda \to p$ in the strong operator topology on A^{**}.

Lemma 1. *Suppose f and g are inequivalent pure states on a C*-algebra $\mathcal{A}$, (a_λ) excises f, and (b_λ) excises g. Also assume that (b_λ) is decreasing. Then $\|a_\lambda x b_\lambda\| \to 0$ for all $x \in \mathcal{A}$.*

Proof: Note that we are assuming (a_λ) and (b_λ) have the same index set. This is easy to arrange by replacing two different index sets with their product. Let q denote the rank one projection in $\mathcal{A}^{**}$ that supports the pure state g. Suppose the lemma is false for some $x \in \mathcal{A}$. Then, multiplying x by a large enough positive number and passing to a subnet, we can assume that $\|a_\lambda x b_\lambda\| > 1$ for all λ.

We first show that $f(xb_\kappa x^*) \searrow 0$. In $\mathcal{A}^{**}$, $xb_\kappa x^* \searrow xqx^*$. Let y denote the central cover of q in $\mathcal{A}^{**}$. Since g is pure, $y\mathcal{A}^{**}$ is isomorphic to some $B(H)$. Thus, f must be 0 on $y\mathcal{A}^{**}$ lest it be equivalent to g. Therefore,

$$f(xb_\kappa x^*) \searrow f(xqx^*) = f(xyqx^*) = f(y(xqx^*)) = 0\,.$$

Fix κ_0 so that $f(xb_{\kappa_0}x^*) < 1/2$, and note that because (a_λ) excises f, there exists λ_0 such that $\lambda \geq \lambda_0$ implies $\|a_\lambda x b_{\kappa_0} x^* a_\lambda - f(xb_{\kappa_0}x^*)a_\lambda^2\| < 1/2$. Choose λ so that $\lambda \geq \lambda_0$ and $\lambda \geq \kappa_0$. Using $b_\lambda \leq b_{\kappa_0}$, we have

$$\begin{aligned}
\|a_\lambda x b_\lambda\|^2 &\leq \|a_\lambda x b_\lambda^{1/2}\|^2 = \|a_\lambda x b_\lambda x^* a_\lambda\| \\
&\leq \|a_\lambda x b_{\kappa_0} x^* a_\lambda\| \\
&\leq \|a_\lambda x b_{\kappa_0} x^* a_\lambda - f(xb_{\kappa_0}x^*)a_\lambda^2\| + f(xb_{\kappa_0}x^*)\|a_\lambda^2\| \\
&< 1/2 + 1/2 = 1\,,
\end{aligned}$$

a contradiction.

Let θ be an action of a discrete group G on a unital C*-algebra $\mathcal{A}$. The reduced crossed product $\mathcal{A} \times_r G$ can be defined as the C*-algebra that acts on the right Hilbert module $l^2(G; \mathcal{A})$ and is generated by (*i*) for each $x \in \mathcal{A}$, the multiplication operator M_x defined by $M_x\phi(g) = \theta_{g^{-1}}(x)\phi(g)$ and (*ii*) for each $h \in G$, the translation operator T_h defined by $T_h\phi(g) = \phi(h^{-1}g)$. Observe that the map $x \mapsto M_x$ isomorphically embeds $\mathcal{A}$ in $\mathcal{A} \times_r G$, so we can regard $\mathcal{A}$ as a subalgebra of $\mathcal{A} \times_r G$ by identifying x with M_x. Denote the identity of G by e.

Theorem 2. *Let $\mathcal{A}$ be a unital C*-algebra, let f be a pure state on $\mathcal{A}$, and let θ be an action of a discrete group G on $\mathcal{A}$. Embed $\mathcal{A}$ in $\mathcal{A} \times_r G$ in the manner just described. Then f has a unique state extension to $\mathcal{A} \times_r G$ if and only if f is inequivalent to $f \circ \theta_g$ for all $g \neq e$.*

Proof: ($\Rightarrow$) Suppose that for some $g \in G$ not the identity, f is equivalent to $f \circ \theta_g$; we must show that f does not extend uniquely to the crossed product. Let $u \in A$ be a unitary such that $f = u^*(f \circ \theta_g)u$ (ref. 6, proposition 3.13.4). Let $\pi : \mathcal{A} \to \mathcal{B}(H_f)$ be the GNS representation associated to f, and let $\xi \in H_f$ be the image of the unit of $\mathcal{A}$, so that $f(y) = \langle \pi(y)\xi, \xi \rangle$ for all $y \in \mathcal{A}$. Let $b \in \mathcal{A}$ be any positive, norm-one element such that $f(b) = 1$; then $\pi(b)(\xi) = \xi$. Furthermore, $\theta_g(u^*bu)$ is also a positive, norm-one element such that $f(\theta_g(u^*bu)) = f(b) = 1$, so the same is true of this element, and it follows that $\pi(b\theta_g(u^*bu))(\xi) = \xi$. Hence $\|b\theta_g(u^*bu)\| = 1$.

Now let $x = \theta_g(u^*)$. Let $b \in \mathcal{A}$ be any positive, norm-one element such that $f(b) = 1$ and let $y \in \mathcal{A}$. We will show that $\|y - b(xT_g)b\| \geq 1$ (computing the norm in the crossed product), which implies nonunique extension by ref. 19, theorem 3.2. Observe first that

$$b(xT_g)b = bx\theta_g(b)T_g = b\theta_g(u^*bu)\theta_{g^{-1}}(u)T_g = cT_g$$

where $c \in \mathcal{A}$ has norm 1. [It is the product of $b\theta_g(u^*bu)$, which we showed above has norm 1, with the unitary $\theta_{g^{-1}}(u)$.] Now to estimate the norm of $y - b(xT_g)b = y - cT_g$, consider the element δ_e of $l^2(G; \mathcal{A})$ that satisfies $\delta_e(e) = 1_{\mathcal{A}}$ (the unit of $\mathcal{A}$) and $\delta_e(h) = 0$ for all $h \neq e$. Since $g \neq e$, we have $(y - cT_g)(\delta_e) = \phi$ where $\phi(e) = y$, $\phi(g) = -\theta_{g^{-1}}(c)$, and $\phi(h) = 0$ for all other h. The norm of δ_e in $l^2(G; \mathcal{A})$ is $\|\Sigma_h \delta_e(h)^*\delta_e(h)\|_{\mathcal{A}}^{1/2} = \|1_{\mathcal{A}}\|_{\mathcal{A}}^{1/2} = 1$ and the norm of ϕ is

$$\|\sum_h \phi(h)^*\phi(h)\|_{\mathcal{A}}^{1/2} = \|y^*y + \theta_{g^{-1}}(c)^*\theta_{g^{-1}}(c)\|_{\mathcal{A}}^{1/2} \geq 1$$

since $\|\theta_{g^{-1}}(c)\| = \|c\| = 1$. Thus the norm of $y - cT_g$ is at least 1, as we needed to show.

($\Leftarrow$) Suppose f is not equivalent to $f \circ \theta_g$ for any $g \neq e$. To verify that f has a unique extension to the crossed product, according to (ref. 19, theorem 3.2), we must, for every element z of the crossed product and every $\varepsilon > 0$, find $x \in \mathcal{A}$ and a positive norm-one element $b \in \mathcal{A}$ such that $f(b) = 1$ and $\|x - bzb\| \leq \varepsilon$. It is sufficient to accomplish this only for a dense set of elements z, so let $z = \Sigma_g x_g T_g$ where each x_g is an element of $\mathcal{A}$ and $x_g \neq 0$ for only finitely many g. (Such sums are clearly dense in $\mathcal{A} \times_r G$.)

Let (a_λ) be a decreasing net in $\mathcal{A}$ that excises f and satisfies $f(a_\lambda) = 1$ for all λ (ref. 18, proposition 2.2). We claim that $\|f(x_e)a_\lambda^2 - a_\lambda z a_\lambda\| \to 0$, which will complete the proof. Observe first that $\|f(x_e)a_\lambda^2 - a_\lambda x_e a_\lambda\| \to 0$ since (a_λ) excises f. Since z is a finite sum and $x_e = x_e T_e$ we now need only show that $\|a_\lambda(x_g T_g)a_\lambda\| \to 0$ for each $g \neq e$. But $a_\lambda(x_g T_g)a_\lambda = (a_\lambda x_g b_\lambda)T_g$ where $b_\lambda = \theta_g(a_\lambda)$ and (b_λ) is a decreasing net that excises $f \circ \theta_{g^{-1}}$. By hypothesis, f and $f \circ \theta_{g^{-1}}$ are inequivalent, so Lemma 1 now implies $\|a_\lambda x_g b_\lambda\| \to 0$. Thus $\|(a_\lambda x_g b_\lambda)T_g\| \to 0$, as desired.

Note that in the proof of the reverse direction of Theorem 2, the fact that $z = \Sigma x_g T_g$ can be "compressed" to x_e implies that the unique extension f' of f satisfies $f'(z) = f(x_e)$.

By ref. 20, the pure state space of any simple, separable C*-algebra $\mathcal{A}$ is homogeneous: given any two inequivalent pure states f and g, there is an automorphism ω of $\mathcal{A}$ such that $f = g \circ \omega$. In the following corollary we need an even more powerful "strong transitivity" result that was proven in ref. 21, theorem 7.5 for a certain class of simple, separable C*-algebras. By combining the methods of the two papers, that result can be achieved for all simple, separable C*-algebras (A. Kishimoto, personal communication).

The result we need states the following: *Suppose $\mathcal{A}$ is a simple, separable C*-algebra and (π_n) and (ρ_n) are sequences of irreducible representations such that the π_n are mutually inequivalent, as are the ρ_n. Then there is an automorphism ω of $\mathcal{A}$ such that π_n is equivalent to $\rho_n \circ \omega$ for all n.* It can be proven by first replacing

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every result in ref. 20 involving a single pure state f (or a pair of pure states f and g) with a corresponding result involving a finite set of mutually inequivalent pure states f_i (or a pair of such sets, with, for each i, f_i and g_i related in the same way that f and g were). The only difference in the proofs will be that every application of Kadison's transitivity theorem will now require the n-fold version for a finite family of mutually inequivalent pure states (ref. 7, theorem 1.21.16). This achieves a proof of ref. 21, theorem 7.3, for any simple separable C*-algebra, which can be converted into a proof of the result we need in the same way that this is done in ref. 21, theorem 7.5.

Corollary 3. *Let $\mathcal{A}$ be a simple, separable, unital C*-algebra and let f and g be inequivalent pure states on $\mathcal{A}$. Then there is a simple, separable, unital C*-algebra $\mathcal{B}$ that unitally contains $\mathcal{A}$ such that f and g have unique state extensions to $\mathcal{B}$ and these extensions are equivalent.*

Proof: Observe first that if f_1 and f_2 are equivalent pure states on $\mathcal{A}$ and f_1 has a unique state extension to $\mathcal{B}$, then so does f_2. Indeed, if $u \in \mathcal{A}$ is a unitary such that $f_1 = u^* f_2 u$ then u also belongs to $\mathcal{B}$ and conjugates the set of extensions of f_1 with the set of extensions of f_2.

It follows from the result quoted before this corollary that given any sequence (π_n) $(n \in \mathrm{Z})$ of inequivalent irreps, there is an automorphism ω of $\mathcal{A}$ such that $\pi_n \circ \omega$ is equivalent to π_{n+1} for all n. Now, since A is simple and separable and has inequivalent pure states it cannot be type I, and therefore it has uncountably many inequivalent irreps (ref. 6, corollary 6.8.5). Let (π_n) be any sequence of mutually inequivalent irreps such that π_1 and π_2 are the GNS irreps arising from f and g, and find an automorphism ω as above. Then g is equivalent to $f \circ \omega$, and neither f nor g is equivalent to itself composed with any nonzero power of ω.

Define $\theta : \mathrm{Z} \to \mathrm{Aut}(\mathcal{A})$ by $\theta_n = \omega^n$ and let $\mathcal{B}$ be the crossed product of $\mathcal{A}$ by this action of Z. By Theorem 2, f and g extend uniquely to $\mathcal{B}$, and therefore so do all pure states equivalent to g, in particular $f \circ \omega$, by the comment at the start of this proof. Now if f' and $(f \circ \omega)'$ are the unique extensions of f and $f \circ \omega$ to $\mathcal{B}$, then

$$f'(T_1(\sum x_n T_n)T_{-1}) = f'(\sum \omega(x_n)T_n) = f(\omega(x_0)) = (f \circ \omega)'(\sum x_n T_n)$$

for any finite sum $\Sigma x_n T_n$ using the comment following Theorem 2. This shows that $T_1 f' T_{-1} = (f \circ \omega)'$, and hence f' is equivalent to $(f \circ \omega)'$. Using the first paragraph again, we see that the unique extensions of f and g to $\mathcal{B}$ are equivalent.

$\mathcal{B}$ is clearly separable and unital, and it unitally contains $\mathcal{A}$. It is simple by ref. 22, theorem 3.1.

We thank A. Kishimoto for a simplification in the above proof.

The Counterexample

A subset S of $\aleph_1$ is said to be *closed* if for every countable $S_0 \subset S$ we have $\sup S_0 \in S$. It is *unbounded* if for every $\alpha \in \aleph_1$ there exists $\beta \in S$ such that $\beta > \alpha$.

We call a nested transfinite sequence of C*-algebras $(\mathcal{A}_\alpha)$ *continuous* if for every limit ordinal α we have $\mathcal{A}_\alpha = \overline{\cup_{\beta<\alpha}\mathcal{A}_\beta}$.

Lemma 4. *Let $(\mathcal{A}_\alpha)$, $\alpha < \aleph_1$, be a continuous nested transfinite sequence of separable C*-algebras and set $\mathcal{A} = \cup\mathcal{A}_\alpha$. Then $\mathcal{A}$ is a C*-algebra, and if f is a pure state on $\mathcal{A}$ then $\{\alpha : f$ restricts to a pure state on $\mathcal{A}_\alpha\}$ is closed and unbounded.*

Proof: We observe first that $\mathcal{A}$ is automatically complete. For any sequence $(x_n) \subset \mathcal{A}$ we can find indices α_n such that $x_n \in \mathcal{A}_{\alpha_n}$, and then $(x_n) \subset A_\alpha$ for $\alpha = \sup\alpha_n$. Thus if (x_n) is Cauchy, its limit belongs to $\mathcal{A}_\alpha$ and hence to $\mathcal{A}$. This shows that $\mathcal{A}$ is a C*-algebra.

Let f be a pure state on $\mathcal{A}$ and let $S = \{\alpha : f$ restricts to a pure state on $\mathcal{A}_\alpha\}$. First we verify that S is closed. Suppose $S_0 \subset S$ is countable and let $\alpha = \sup S_0$. If $f|_{\mathcal{A}_\alpha}$ is not pure then we can write $f|_{A_\alpha} = (f_1 + f_2)/2$, where f_1 and f_2 are distinct states on $\mathcal{A}_\alpha$. Now $\cup_{\beta\in S_0} \mathcal{A}_\beta$ is dense in $\mathcal{A}_\alpha$ by continuity (unless $\alpha \in S_0$, when this is true vacuously). Thus there exists $\beta \in S_0$ such that $f_1|_{\mathcal{A}_\beta} \neq f_2|_{\mathcal{A}_\beta}$. But then $f|_{\mathcal{A}_\beta} = (f_1|_{\mathcal{A}_\beta} + f_2|_{\mathcal{A}_\beta})/2$ contradicts the fact that $f|_{\mathcal{A}_\beta}$ is pure. We conclude that $f|_{\mathcal{A}_\alpha}$ is pure and hence that S is closed.

Next observe that there is a sequence (x_n) in $\mathcal{A}$ such that $|f(x_n)|/\|x_n\| \to 1$. Then $(x_n) \subset \mathcal{A}_\alpha$ for some $\alpha < \aleph_1$, so that the restriction of f to $\mathcal{A}_\beta$ has norm one, and hence is a state, for all $\beta \geq \alpha$. Thus, without loss of generality, in proving unboundedness we can assume the restriction of f to any $\mathcal{A}_\alpha$ is a state.

Let $\alpha < \aleph_1$. We first claim that for any $x \in \mathcal{A}_\alpha$, for sufficiently large $\beta \geq \alpha$ we have

$$f|_{\mathcal{A}_\beta} = (f_1 + f_2)/2 \Rightarrow f_1(x) = f_2(x)$$

whenever f_1 and f_2 are states on $\mathcal{A}_\beta$. That is, for each $x \in \mathcal{A}_\alpha$ there exists $\alpha' \geq \alpha$ such that the above holds for all $\beta \geq \alpha'$. If the displayed condition holds for all states f_1 and f_2 on $\mathcal{A}_\beta$, then we say that $f|_{\mathcal{A}_\beta}$ is *pure on x.*

Suppose the claim fails for some $x \in \mathcal{A}_\alpha$. Then there exist $\varepsilon > 0$ and an unbounded set $T \subset \aleph_1$ together with states f_1^β and f_2^β on $\mathcal{A}_\beta$ for all $\beta \in T$, such that

$$\mathrm{f}|_{\mathcal{A}_\beta} = (\mathrm{f}_1^\beta + \mathrm{f}_2^\beta)/2 \text{ and } |f_1^\beta(x) - f(x)| \geq \varepsilon. \qquad [*]$$

(If no such ε and T existed, then for each $n \in \mathrm{N}$ we could find $\alpha_n < \aleph_1$ such that for any $\beta \geq \alpha_n$, no states f_1^β and f_2^β on $\mathcal{A}_\beta$ satisfy * with $\varepsilon = 1/n$. Then $f|_{\mathcal{A}_\beta}$ would be pure on x for all $\beta \geq \sup \alpha_n$, contradicting our assumption that the claim fails for x.) Now let $\mathcal{U}$ be an ultrafilter on T that contains the set $\{\beta \in T : \beta > \beta_0\}$ for each $\beta_0 < \aleph_1$, and define states g_1 and g_2 on $\mathcal{A}$ by $g_1 = \lim_u f_1^\beta$ and $g_2 = \lim_u f_2^\beta$, where the limits are taken pointwise on elements of $\mathcal{A}$. Then $f = (g_1 + g_2)/2$ and $|g_1(x) - f(x)| \geq \varepsilon$ hence $f \neq g_1$, contradicting purity of f. (This argument could also be carried out by using universal nets.) This establishes the claim.

Now for any $\alpha < \aleph_1$, since $\mathcal{A}_\alpha$ is separable we can find a dense sequence $(x_n) \subset \mathcal{A}_\alpha$, and the claim implies that for sufficiently large β, $f|_{\mathrm{A}_\beta}$ is pure on every x_n. By density, $f|_{\mathcal{A}_\beta}$ is then pure on every $x \in \mathcal{A}_\alpha$. Let α^* be the least ordinal larger than α such that $\beta \geq \alpha^*$ implies that $f|_{\mathcal{A}_\beta}$ is pure on every $x \in \mathcal{A}_\alpha$.

Finally, to see that S is unbounded, fix $\alpha < \aleph_1$. Let $\alpha_1 = \alpha^*$, $\alpha_2 = \alpha^{**}$, etc., and let $\beta = \sup\alpha_n$. Then $f|_{\mathcal{A}_\beta}$ is pure on every $x \in \cup\mathcal{A}_{\alpha_n}$, and hence by continuity $f|_{\mathcal{A}_\beta}$ is a pure state on $\mathcal{A}_\beta$. This shows that $\beta \in S$, as desired.

A subset of $\aleph_1$ is *stationary* if it intersects every closed unbounded subset of $\aleph_1$. We require the following version of the diamond principle ($\diamond$): there exists a transfinite sequence of functions $h_\alpha : \alpha \to \aleph_1$ $(\alpha < \aleph_1)$ such that for any function $h : \aleph_1 \to \aleph_1$ the set $\{\alpha : h|_\alpha = h_\alpha\}$ is stationary (see ref. 13, 22.20, or ref. 14, exercise II.51).

Let $S(\mathcal{A})$ denote the set of states on a C*-algebra $\mathcal{A}$.

Theorem 5. *Assume $\diamond$. Then there is a counterexample to Naimark's problem that is generated by $\aleph_1$ elements.*

Proof: Let (h_α) be a transfinite sequence of functions which verifies $\diamond$ in the form given above. For $\alpha < \aleph_1$ we recursively construct a continuous nested transfinite sequence of simple separable unital C*-algebras $\mathcal{A}_\alpha$, all with the same unit; pure states f_α on $\mathcal{A}_\alpha$ with the property that $\alpha < \beta$ implies f_β is the unique state extension of f_α; and injective functions $\phi_\alpha : S(\mathcal{A}_\alpha) \to \aleph_1$ as follows. Let $\mathcal{A}_0$ be any simple, separable, infinite dimensional, unital C*-algebra and let f_0 be any pure state on $\mathcal{A}_0$. Since $\diamond$ implies the continuum hypothesis (see ref. 13 or 14), $S(\mathcal{A}_0)$ has cardinality at most $\aleph_1$, so there exists an injective function from $S(\mathcal{A}_0)$ into $\aleph_1$; let ϕ_0 be any such function.

To proceed from stage α of the construction to stage $\alpha + 1$ when α is a limit ordinal, first check whether there is a pure state g_α on $\mathcal{A}_\alpha$, not equivalent to f_α, such that $h_\alpha(\beta) = \phi_\beta(g_\alpha|_{\mathcal{A}_\beta})$ for all

$\beta < \alpha$. (By injectivity of all ϕ_β, there is at most one such g_α.) If so, let $\mathcal{A}_{\alpha+1}$ be the C*-algebra $\mathcal{B}$ given by Corollary 3 with $f = f_\alpha$ and $g = g_\alpha$. Let $f_{\alpha+1}$ be the unique extension of f_α to $\mathcal{A}_{\alpha+1}$, and as above let $\phi_{\alpha+1}$ be any injective function from $S(\mathcal{A}_{\alpha+1})$ into $\aleph_1$. If there is no such state g_α, and whenever α is a successor ordinal (or $\alpha = 0$), let $\mathcal{A}_{\alpha+1} = \mathcal{A}_\alpha$, $f_{\alpha+1} = f_\alpha$, and $\phi_{\alpha+1} = \phi_\alpha$. At limit ordinals α, let $\mathcal{A}_\alpha = \overline{\cup_{\beta<\alpha} \mathcal{A}_\beta}$ (it is standard that $\mathcal{A}_\alpha$ will be simple, given that each $\mathcal{A}_\beta$ is simple), define f_α by requiring $f_\alpha|_{\mathcal{A}_\beta} = f_\beta$ for all $\beta < \alpha$ (f_α will be pure, by the argument in the second paragraph of the proof of Lemma 4), and as before let ϕ_α be any injective function from $S(\mathcal{A}_\alpha)$ into $\aleph_1$.

Let $\mathcal{A} = \cup_{\alpha<\aleph_1} \mathcal{A}_\alpha$ and define $f \in S(\mathcal{A})$ by $f|_{A_\alpha} = f_\alpha$. Here f is well defined since f_β is an extension of f_α whenever $\alpha < \beta$. Also, f is a pure state, again by the reasoning in the second paragraph of the proof of Lemma 4 coupled with the fact that each f_α is pure. We claim that every pure state g on $\mathcal{A}$ is equivalent to f. To see this, it is enough to verify that $g|_{\mathcal{A}_\alpha}$ is equivalent to f_α for some α; then since f_α extends uniquely to $\mathcal{A}$ the same must be true of $g|_{\mathcal{A}_\alpha}$ (see the first paragraph of the proof of Corollary 3), and the unitary in $\mathcal{A}_\alpha$ that implements the equivalence of f_α and $g|_{\mathcal{A}_\alpha}$ must then also implement an equivalence between f and g. Now define $h : \aleph_1 \to \aleph_1$ by setting $h(\alpha) = \phi_\alpha(g|_{\mathcal{A}_\alpha})$. Let S be the set of limit ordinals α such that $g|_{\mathcal{A}_\alpha}$ is pure. According to Lemma 4, S is closed and unbounded. (The intersection of any two closed unbounded sets is always closed and unbounded.) Therefore, by $\diamondsuit$ there exists a limit ordinal α such that $g|_{\mathcal{A}_\alpha}$ is pure and $h_\alpha = h|_\alpha$, i.e.,

$$h_\alpha(\beta) = \phi_\beta(g|_{\mathcal{A}_\beta})$$

for all $\beta < \alpha$. If $g|_{\mathcal{A}_\alpha}$ is equivalent to f_α then we are done, and otherwise the construction at stage α guarantees that $f_{\alpha+1}$ is equivalent to the unique extension of $g|_{\mathcal{A}_\alpha}$ to $\mathcal{A}_{\alpha+1}$, which must be $g|_{\mathcal{A}_{\alpha+1}}$. This completes the proof that f and g are equivalent.

Finally, $\mathcal{A}$ is infinite dimensional and unital, so it cannot be isomorphic to any $\mathcal{K}(H)$.

We conclude by observing that if the continuum hypothesis fails then there is no counterexample to Naimark's problem that is generated by $\aleph_1$ elements.

Proposition 6. *Let $\mathcal{A}$ be a counterexample to Naimark's problem. Then $\mathcal{A}$ cannot be generated by fewer than $2^{\aleph_0}$ elements.*

Proof: Since $\mathcal{A}$ cannot be type I, it follows that there is a subalgebra $\mathcal{B}$ of $\mathcal{A}$ and a surjective *-homomorphism of $\mathcal{B}$ onto the Fermion algebra $\mathcal{F} = \otimes_1^\infty M_2(\mathrm{C})$ (ref. 6, corollary 6.7.4). Let f_1 and f_2 be the pure states on $M_2(\mathrm{C})$ defined by

$$f_1\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11} \text{ and } f_2\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{22}\,.$$

Then for any sequence of indices $i_n \in \{1, 2\}$ the product state $f = \otimes_n f_{i_n}$ is a pure state on $\mathcal{F}$. Moreover, $\|f - f'\| = 2$ for any two distinct states of this form. Lifting to $\mathcal{B}$ and extending to $\mathcal{A}$, we obtain a family of $2^{\aleph_0}$ pure states on $\mathcal{A}$, the distance between any two of which is 2. As $\mathcal{A}$ is a counterexample to Naimark's problem, there exists a family of $2^{\aleph_0}$ unitaries in the unitization $\tilde{\mathcal{A}}$ of $\mathcal{A}$ that conjugate this family of pure states to a single (arbitrary) pure state g on $\mathcal{A}$. Uniform separation of the pure states implies uniform separation of the conjugating unitaries: if $g = u_1^* f_1 u_1 = u_2^* f_2 u_2$, then for all $x \in \mathcal{A}$

$$\begin{aligned} |f_1(x) - f_2(x)| &= |f_1(x - u^*xu)| \\ &\le |f_1(x - u^*x)| + |f_1(u^*x - u^*xu)| \\ &\le 2\|1_A - u\|\|x\|\,, \end{aligned}$$

where $u = u_2^* u_1$, so $\|f_1 - f_2\| \le 2\|1_{\mathcal{A}} - u\| = 2\|u_1 - u_2\|$. Thus $\tilde{\mathcal{A}}$, and hence $\mathcal{A}$, cannot contain a dense set with fewer than $2^{\aleph_0}$ elements, and consequently it cannot be generated by fewer than $2^{\aleph_0}$ elements.

Corollary 7. *If the continuum hypothesis fails, then no C*-algebra generated by $\aleph_1$ elements is a counterexample to Naimark's problem. The existence of a counterexample to Naimark's problem which is generated by $\aleph_1$ elements is independent of ZFC.*

The first assertion of Corollary 7 follows immediately from Proposition 6, and the second assertion follows from Theorem 5 coupled with the first assertion, together with the fact that both diamond and the negation of the continuum hypothesis are consistent with ZFC (assuming ZFC is consistent).

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