From the representation theory of vertex operator algebras to modular tensor categories in conformal field theory

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Two-dimensional conformal quantum field theory (CFT) has inspired an immense amount of mathematics and has interacted with mathematics in very rich ways, in great part through the mathematically dynamic world of string theory. One notable example of this interaction is provided by Verlinde's conjecture: E. Verlinde (1) conjectured that certain matrices formed by numbers called the "fusion rules" in a "rational" CFT are diagonalized by the matrix given by a certain natural action of a fundamental modular transformation (essentially, a certain distinguished element of the group of two-by-two matrices of determinant one with integer entries). His conjecture led him to the "Verlinde formula" for the fusion rules and, more generally, for the dimensions of spaces of "conformal blocks" on Riemann surfaces of arbitrary genera. A great deal of progress has been achieved in interpreting and proving Verlinde's (physical) conjecture and the Verlinde formula in mathematical settings, in the case of the Wess–Zumino–Novikov-Witten models in CFT, which are based on affine Lie algebras. On the other hand, Moore and Seiberg (2, 3) showed, on a physical level of rigor, that the general form of the Verlinde conjecture is a consequence of the axioms for rational CFTs, thereby providing a conceptual understanding of the conjecture. In the process, they formulated a CFT analogue, later termed "modular tensor category" (discussed in refs. 4 and 5) by I. Frenkel, of the classical notion of tensor category for representations of (modules for) a group or a Lie algebra. It remained a very deep problem to construct, in a mathematical as opposed to physical sense, structures ("theories") satisfying these axioms for rational CFT. These axioms are, in fact, much stronger than the Verlinde conjecture and modular tensor category structure, and, indeed, the mathematical construction of CFTs (as opposed to the physical assumption that they should exist) is a very rich field of study to which many mathematicians have contributed. In this issue of PNAS, Huang (6) announces a (mathematical) proof of the Verlinde conjecture in a very general form, along with two notable consequences: the rigidity and modularity of a previously constructed tensor category. This work, along with results used in this work, includes, in particular, the (mathematical) construction of a significant portion of CFT—structures that actually do satisfy the axioms—using the representation theory of vertex operator algebras.

Huang (6) invokes a great deal of earlier work in vertex (operator) algebra theory. The mathematical foundation of CFT may be viewed as resting on the theory of vertex operator algebras (ref. 7; see also ref. 8), which reflect the physical features codified by Belavin *et al.* (9). Mathematically, vertex operator algebra theory is extremely rich. For the work discussed here, one needs the representation theory of vertex operator algebras, especially a tensor product theory for modules for a suitable vertex operator algebra. In classical tensor product theories for modules for a group or for a suitable algebra such as a Lie algebra, one automatically has the tensor product vector space available, and one endows it with tensor product module structure by means of a natural coproduct operation. A module map from the tensor product of two modules to a third module then amounts to an "intertwining operator" satisfying a natural condition coming from the group or algebra actions on the three modules.

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However, vertex operator algebra theory is imbued with considerable "nonclassical" subtleties, intimately related to the nonclassical nature of string theory in physics, and to construct a tensor product theory of modules for a vertex operator algebra, one is forced to proceed "backwards": First, one defines suitable "intertwining operators" (3, 10) among triples of modules. The dimensions of the spaces of these intertwining operators are the fusion rules referred to above. Then one has to construct a tensor product theory that "implements" these intertwining operators. After work of Kazhdan–Lusztig (11) for certain structures based on affine Lie algebras, a tensor product theory for modules for a suitable, general vertex operator algebra was constructed in a series of papers summarized in ref. 12. Elaborate use of "formal calculus" (discussed in ref. 8) was required in this work. The main paper in this series (13) establishes Huang's associativity theorem, which leads quickly to the (categorical) coherence of the resulting braided tensor category. In CFT terminology, this associativity theorem asserts the existence and associativity of the operator product expansion for intertwining operators, an assertion that was a key assumption (not theorem) in ref. 3. In addition to braided tensor category structure, this series of papers constructs the much richer "vertex tensor category" structure, which involves the conformal-geometric structure established in ref. 14, on the module category of a suitable vertex operator algebra.

A fundamental theorem establishing natural modular transformation properties of "characters" of modules for a suitable vertex operator algebra was proved by Zhu (15). Requiring all of the theory mentioned here, as well as results in refs. 16–18, Huang formulates a general, mathematically precise, statement of the Verlinde conjecture in the framework of the theory of vertex operator algebras. Assuming only such purely algebraic, natural hypotheses as simplicity of the vertex operator algebra, complete reducibility of suitable modules, natural grading restrictions, and cofiniteness, hypotheses that are relatively easily checked and have indeed been previously verified in a wide range of important families of examples, Huang sketches his proof (see ref. 6). The proof is heavily based on the results of his recent papers (19, 20), in which natural duality and mod-

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ular invariance properties for genus-zero and genus-one multipoint correlation functions constructed from intertwining operators for a vertex operator algebra satisfying the general hypotheses are established; the multiple-valuedness of the multipoint correlation functions leads to considerable subtleties that had to be handled analytically and geometrically, rather than just algebraically. The strategy of the proof reflects the pattern of refs. 2 and 3; in fact, the main work is to establish two formulas of Moore and Seiberg that they had derived from strong assumptions: the axioms for rational CFT. The difficulties lie in the sequence of mathematical developments briefly mentioned here.

As has been the case with many other major developments in the mathematical study of string theory and conformal field theory over the years, it is to be expected that the methods used by Huang (6) will have further consequences. In fact, this tensor category theory has already been applied to a variety of fields in mathematics and physics, including string theory or M-theory, in particular, D-branes. The insight that continues to flow from the combined and respective efforts of many physicists and mathematicians in this remarkable age of string theory and its mathematical counterparts will surely produce new surprises.

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