# A note concerning the Lighthill ''sandwich model'' of tropical cyclones

G. I. Barenblatt[†‡], A. J. Chorin[†], and V. M. Prostokishin[§]

[†]Department of Mathematics, University of California at Berkeley and Lawrence Berkeley National Laboratory, Berkeley, CA 94720-3840; and [§]Institute of Oceanology, Russian Academy of Sciences, 36 Nakhimov Prosp., Moscow 117218, Russia

Contributed by A. J. Chorin, June 22, 2005

**The basic element of Lighthill's ''sandwich model'' of tropical cyclones is the existence of ''ocean spray,'' a layer intermediate between air and sea made up of a cloud of droplets that can be viewed as a ''third fluid.'' We propose a mathematical model of the flow in the ocean spray based on a semiempirical turbulence theory and demonstrate that the availability of the ocean spray over the waves in the ocean can explain the tremendous acceleration of the wind as a consequence of the reduction of the turbulence intensity by droplets. This explanation complements the thermodynamic arguments proposed by Lighthill.**

ocean spray | turbulence in stratified fluids

In his posthumously published work (1), Lighthill emphasized ''the need to fill gaps in knowledge about the ocean spray at extreme wind speeds.'' Lighthill himself concentrated on the thermodynamic issues in the phenomena: the variation of the bulk flow energy due to the evaporation of droplets in the ocean spray and the subsequent cooling of the atmosphere. A recent review of the physics of tropical cyclone motion can be found in ref. 2, where, by the way, the fundamental work (1) is not referenced.

Our approach is different and complements Lighthill's. We consider the modification of the balance of turbulent energy of the flow in ocean spray by the suspension of droplets. Furthermore, we demonstrate, by using the general theory of turbulent transfer of suspended heavy particles developed by A. N. Kolmogorov and G.I.B. (see pp. 301–306 in ref. 3 and the references cited there) that the presence of large water droplets in the ocean spray leads to a significant reduction of the turbulence intensity in the flow and consequently to a sharp flow acceleration. The cooling due to the evaporation of the droplets makes an additional contribution to the reduction of turbulence, to the stabilization of the flow stratification, and to the flow acceleration.

## Basic Mathematical Model

We concentrate here on a single effect: flow acceleration in an ocean spray that carries large water droplets, leaving aside the influence of the Coriolis force and the cooling effects due to the evaporation of the droplets. We demonstrate that flow acceleration by droplets is very significant by itself. The effects of air cooling by evaporation and of the Coriolis force can be included easily in numerical models.

Thus, we consider a steady horizontally homogeneous flow of the air–droplets mixture, so that all components of the field, including the velocity $u$, the volume concentration of the droplets $s$, etc., depend only on the vertical coordinate $z$.

The phenomenon is sketched in Fig. 1. We average the pattern over the horizontal coordinate. The waves, whose average height is $z_W$, form the water–air boundary. After averaging the wave crests form a horizontal line $z = z_W$. The waves break (think of the famous painting of Hokusai), and on average the jets formed at the crests break up at the height $z = z_0$ creating droplets. We assume in our model that all droplets are identical, and the concentration of the droplets in a thin layer around $z = z_0$ is equal to $s = s_0$. We neglect the reduction in the radius of the droplets

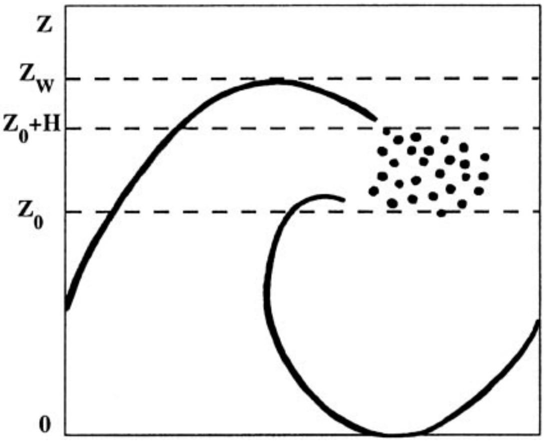


**Fig. 1.** Schematic of ocean spray formation.

due to evaporation and also the coalescence of drops. The droplets above $z = z_0$ are suspended by the air flow; below $z = z_0$ they also are suspended, but part of them is falling into water.

In our analysis of the flow in the ocean spray, we will use the $(\ell, b)$ version of the Kolmogorov–Prandtl semiempirical model of turbulent shear flow (see, e.g., ref. 3, pp. 280–282). In this version, all kinematic characteristics of the turbulence are assumed to be determined by the external turbulence length scale $\ell$ and by the mean kinetic energy of turbulence per unit mass $b$ (in the literature, the last quantity is often denoted by $k$, but we prefer to remain with the original notations of Kolmogorov). As was emphasized by Kolmogorov, this hypothesis as well as the similar hypothesis in the $(b, \varepsilon)$ version ($\varepsilon$ is the dissipation rate of turbulent energy into heat per unit mass) brings a strong simplification to the theory and therefore can be used only for semiquantitative models. Our goal, however, is the construction of such a model of the flow in ocean spray.

The volume and mass concentration of droplets are small, and we assume that the difference between air density and the density of the mixture of air and droplets in ocean spray can be neglected everywhere where this difference is not multiplied by a very large factor: the gravity acceleration $g$. Therefore, the equation of momentum balance of the flow in ocean spray remains the same as in a pure air flow. The equation of mass balance of droplets (we neglect the evaporation and coalescence) represents the equality of the vertical turbulent flux of droplets to the rate of falling of the droplets. The major distinction between the modeling of the flow in the ocean spray and the shear flow of pure air lies in the equation of balance of turbulent energy: a term is added to the balance between the rate of generation of the turbulent energy and the dissipation rate: the rate of turbulent energy dissipation into the suspension of droplets. Here, the difference between the densities of air and

[‡]To whom correspondence should be addressed. E-mail: gibarenblatt@lbl.gov.

the air–droplets mixture becomes of fundamental importance because it is multiplied by the gravity acceleration $g$.

Applying the closure hypotheses of the $(\ell, b)$ version of the Kolmogorov–Prandtl semiempirical theory of turbulence to the equations of balance of momentum, water mass in the droplets, and turbulent energy, we obtain the basic system of equations of the model of the flow in ocean spray (see ref. 3, pp. 301–303)

$$\ell \sqrt{b} \partial_z u = u_*^2 \quad [1]$$

$$\alpha_s \ell \sqrt{b} \partial_z s + as = 0 \quad [2]$$

$$b = \frac{u_*^2}{\gamma^2}(1 - Ko)^{1/2}. \quad [3]$$

Here, $u(z)$ is the mean velocity of the flow at the level $z$, $u_* = \sqrt{\tau/\rho}$, $\tau$ is the tangential stress at the spray–sea boundary, and $\rho$ is the air density. Furthermore, $\alpha_s$ is an analog of the turbulent Prandtl number, which we will assume constant and equal to 2; $\gamma$ is a constant that was evaluated from the experimental data for the flow of a homogeneous fluid, $\gamma \simeq 0.5$; $a$ is the velocity of the free fall of a droplet in the air at rest, $\sigma = (\rho_d - \rho)/\rho$, where $\rho_d$ is water density, $\rho$ is air density, and $\sigma$ is a large number of the order of $10^3$.

The Kolmogorov dimensionless parameter $Ko$

$$Ko = \frac{\sigma g a s}{u_*^2 \partial_z u}, \quad [4]$$

is the basic dimensionless parameter in the model; it is equal to the ratio of the rate of energy dissipation for the suspension of droplets to the rate of the generation of turbulent energy. Clearly, $0 \le Ko < 1$.

To close the system of Eqs. **1–4**, an additional assumption concerning the external length scale of turbulence $\ell$ is needed. One can take this length scale to be the local characteristic length scale of the vortices in turbulent shear flow. (In ref. 4, the basic role of vortices in turbulent flows is presented in detail.) As was demonstrated in our previous work (see, e.g., ref. 5), the Reynolds number-dependent scaling law for the velocity distribution in the wall-bounded shear flow

$$u = u_*\left(\frac{1}{\sqrt{3}} \ln Re + \frac{5}{2}\right)\left(\frac{u_* z}{\nu}\right)^{3/2 \ln Re},$$

in an appropriate interval of distances from the wall has an intermediate asymptotics

$$u = \frac{u_*}{\kappa(Re)} \ln \frac{u_* z}{\nu} + B(Re),$$

where $\nu$ is the fluid kinematic viscosity and $Re$ is the global Reynolds number specific to each shear flow. As $Re$ tends to infinity $\kappa(Re)$ tends to a finite limit, $\kappa = \kappa_\infty \simeq 0.2776\ldots$, whereas $B(Re)$ is unbounded (see also ref. 6, pp. 151–153). Bearing in mind large values of $Re$ in ocean sprays, it is natural to take as an external length scale for ocean spray the following quantity, which is proportional to $u_*/\partial_z u$:

$$\ell = \frac{u_*}{\partial_z u} \gamma \Phi(Ko) = \kappa_\infty \gamma z \Phi(Ko), \quad [5]$$

where the dimensionless factor $\Phi(Ko)$ reflects the possible influence of stratification, and the factor $\gamma \simeq 0.5$ is introduced for the convenience of subsequent normalization. Note that if there are no droplets in the flow, $Ko = 0$, and according to Eqs. **3** and **1**, $b = u_*^2/\gamma^2$, $\Phi(0) = 1$. It seems plausible that the droplets reduce the length scale of vortices; therefore, $\Phi(Ko)$ should be less than one at positive $Ko$.

The formation of vortices begins in the hollows of the waves; therefore, the level $z = 0$ corresponds to the level of the origin of the vortices, which justifies the introduction of the argument $z$ into the relation in Eq. **5**.

The droplets start to form at a height that we denote by $z_0 + h \le z_W$. Above this level the droplets are transferred by wind. Below this level an active process of formation, coalescence, and evaporation of droplets is going on, therefore Eq. **2** is no longer valid: a term on the right-hand side representing the rate of formation of droplets should be added to this equation. We are unable nowadays to evaluate this term; therefore, we simplify the consideration by assuming that the concentration of the droplets is constant in a layer of thickness $h$

$$s(z) = s_0, \quad (z_0 \le z \le z_0 + h). \quad [6]$$

The system in Eqs. **1–4** is solved in the interval $z \ge z_0$. Note that the system in Eqs. **1–4** does not contain the velocity $u(z)$ itself; therefore, the velocity is determined up to an additive constant. We can select this constant assuming that the velocity is equal to zero at the level $z = z_0$, because wind speed above the waves is much larger than the velocity of waves:

$$u(z_0) = 0. \quad [7]$$

The relations in Eqs. **1–7** form the proposed mathematical model of the flow in ocean spray.

APPLIED MATHEMATICS

## Analytic Investigation of the Model

We pass to the dimensionless parameters $B(Z) = b/u_*^2$, $U(Z) = u/u_*$, $z = CZ$, $\ell = CL$, $h = CH$, where $C = u_*^2/\alpha_s \sigma g$. The basic equations take the form

$$L\sqrt{B}\frac{dU}{dZ} = 1, \quad \omega s + L\sqrt{B}\frac{ds}{dZ} = 0$$
$$B = \gamma^{-2}(1 - Ko)^{1/2},$$
$$Ko = -\frac{ds}{dZ}\left(\frac{dU}{dZ}\right)^{-2},$$
$$L = \kappa_\infty \gamma Z \Phi(Ko). \quad [8]$$

The dimensionless parameters that enter the model and determine the flow regimes are

$$s_0, Z_W = \frac{z_W}{u_*^2/\alpha_s \sigma g}, Z_0 = \frac{z_0}{u_*^2/\alpha_s \sigma g}, H = \frac{h}{u_*^2/\alpha_s \sigma g}, \omega = \frac{a}{\alpha_s u_*}. \quad [9]$$

The basic system of Eqs. **1–4** can be reduced to a single first-order ordinary differential equation for the inverse concentration of the droplets $R = 1/s$

$$\frac{R}{\kappa_\infty \omega Z} - \Phi\left(\frac{\omega^2}{dR/dZ}\right)\left(1 - \frac{\omega^2}{dR/dZ}\right)^{1/4} \frac{dR}{dZ} \cdot \frac{1}{\omega^2} = 0. \quad [10]$$

Introducing the function $v(w)$ inverse to the function $w = \Phi(1/v)(1 - 1/v)^{1/4} v$, it is possible to present Eq. **10** in the form

$$\frac{dR}{dZ} = \omega \kappa_\infty v\left(\frac{1}{\omega^2}\frac{dR}{dZ}\right), \quad [11]$$

which can be integrated by quadratures taking into account the boundary condition of Eq. **6**

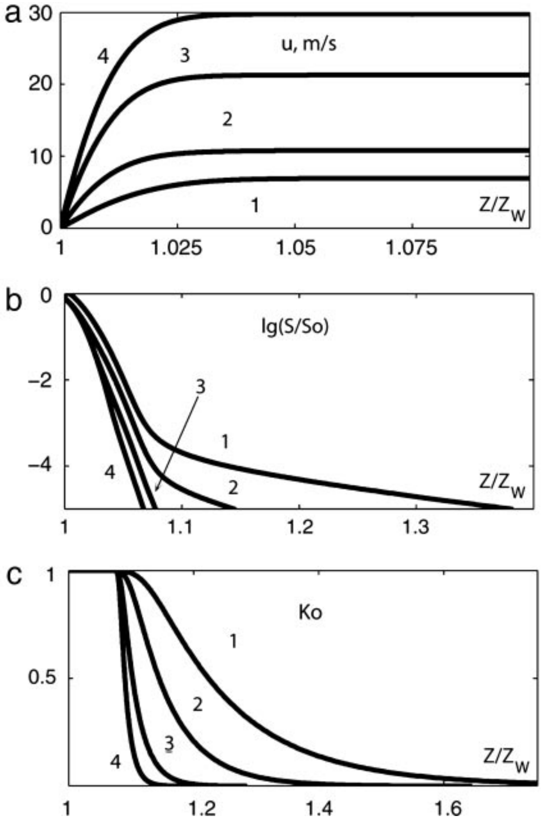


**Fig. 2.** The graphs of velocity (*a*), concentration of droplets (*b*), and Kolmogorov's number (*c*) as functions of height for the following different values of $\omega$: (1) $\omega = 75$, (2) $\omega = 50$, (3) $\omega = 25$, and (4) $\omega = 10$.

$$\ln \frac{Z}{Z_0 + H} = \int_{1/\omega\kappa_\infty Z_0 s_0}^{1/\omega\kappa_\infty Z s} \frac{dX}{\omega v(X) - \kappa_\infty X}. \quad \textbf{[12]}$$

The solution in Eq. **12** allows an effective investigation of the model.

## Discussion

The parameter $\omega = a/\alpha_s u_*$ has a transparent physical meaning: it is the ratio of the free fall velocity to a characteristic value of the velocity of the turbulent fluctuations. Previous investigations of the turbulent transfer of heavy particles concerned mostly the regime of limiting saturation where the particles entrain the main core of the flow. This assumption is possible only for sufficiently small particles. The situation in the ocean spray is different in principle, because, as was emphasized in particular by Lighthill, the water droplets in the ocean spray are large, of the order of 20 $\mu$m and even larger, so that the parameter $\omega$ is substantially larger than one. Conversely, $\kappa_\infty$ is less than one, and the function $w(v)$ is always less than $v$ (this function has real values at $v \geq 1$ only). Therefore, the denominator in the integral of Eq. **12** is always positive, and all solutions for a given value of $\omega$ can be obtained in the ln $Z$, $1/Zs$ plane one from another by a simple shift along the ln $Z$ axis.

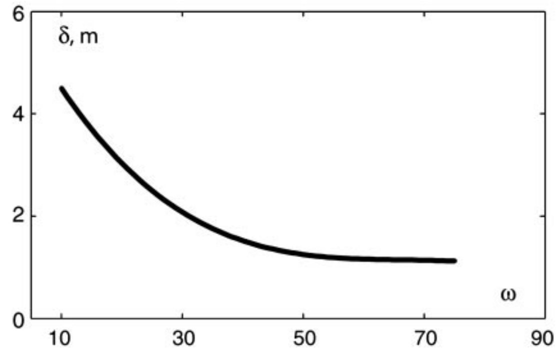


**Fig. 3.** The dependence of ocean spray thickness $\delta$ on $\omega$ determined by the level where $Ko = 0.1$; $h = 10$ cm, $s_0 = 10^{-4}$.

It is easy to show that at large $\omega$, characteristic of the ocean sprays, the concentration $s$ decays with height fast, like $z^{-\omega/\kappa_\infty}$, the Kolmogorov number is also fast decaying, which means that at larger heights the velocity gradient becomes undisturbed by the droplets. This is not the case for the velocity itself: due to the suppression of turbulence in the ocean spray, the flow is strongly accelerated even at moderate pressure gradients, sometimes reaching very high speeds. In Figs. 2 and 3, the results of numerical calculations are presented that illustrate this fact and demonstrate in particular that at large $\omega$, i.e., for the ocean sprays with large droplets, there exists a sharp upper boundary of the ocean spray. An illustrative numerical example is as follows: with standard values of the water and air densities, $u_* = 1$ m/s, $z_W = 10$ m, $z_0 = 9.9$ m, $h = 10$ cm, the wind speed with no droplets at $z = 20$ m is $\approx 4$ m/s, while in the presence of large droplets at $s_0 = 10^{-4}$, $\omega = 75$, the wind velocity is 30 m/s.

## Conclusion

In the present work, we demonstrated that the mechanism of turbulence suppression by water drops in the ocean spray can substantially accelerate the flow so that the speeds of wind characteristic of the strongest hurricanes can be reached. The complete mathematical model, taking into account both thermal effects and Coriolis force, can now be constructed in the form that allows effective numerical calculations. Note that a model of dust storms taking into account the thermal effects was proposed in ref. 7. Furthermore, the effects of particles on the dynamics of tornados can be studied by similar means.

In conclusion, we want to make a comment. Since antiquity, seamen have had barrels of oil on the decks of their vessels and thrown the oil on the sea surface in critical moments of stormy weather. We think that the action of oil was exactly the prevention of the formation of droplets. The turbulence was restored after the oil was dropped, the turbulent drag was increased, and the intensity of the squall was reduced. Possibly hurricanes can be similarly prevented or damped by having airplanes deliver fast decaying harmless surfactants to the right places on the sea surface.

This work is dedicated to the glowing memory of the great mathematician and fluid mechanician Sir James Lighthill, whose suggestion inspired this work. This work was supported by the Director, Office of Science, Advanced Scientific Computing Research, U.S. Department of Energy Contract DE-AC03-76SF00098.

1. Lighthill, J. (1999) *J. Eng. Math.* **35,** 11–42.
2. Chan, J. C. L. (2005) *Annu. Rev. Fluid Mech.* **37,** 99–128.
3. Barenblatt, G. I. (1996) *Scaling, Self-Similarity, and Intermediate Asymptotics* (Cambridge Univ. Press, Cambridge, U.K.).
4. Chorin, A. J. (1993) *Vorticity and Turbulence* (Springer, New York).
5. Barenblatt, G. I., Chorin, A. J. & Prostokishin, V. M. (1997) *Appl. Mech. Rev.* **50,** 413–429.
6. Barenblatt, G. I. (2003) *Scaling* (Cambridge Univ. Press, Cambridge, U.K.).
7. Barenblatt, G. I. & Golitsyn, G. S. (1974) *J. Atmos. Sci.*, 1917–1933.