

A monopole near a black hole

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This contribution is part of the special series of Inaugural Articles by members of the National Academy of Sciences elected on May 3, 2005.

Contributed by Claudio Bunster, May 29, 2007 (sent for review April 24, 2007)

A striking property of an electric charge near a magnetic pole is that the system possesses angular momentum even when both the electric and the magnetic charges are at rest. The angular momentum is proportional to the product of the charges and independent of their distance. We analyze the effect of bringing gravitation into this remarkable system. To this end, we study an electric charge held at rest outside a magnetically charged black hole. We find that even if the electric charge is treated as a perturbation on a spherically symmetric magnetic Reissner–Nordstrom hole, the geometry at large distances is that of a magnetic Kerr–Newman black hole. When the charge approaches the horizon and crosses it, the exterior geometry becomes that of a Kerr–Newman hole, with electric and magnetic charges and with total angular momentum given by the standard value for a charged monopole pair. Thus, in accordance with the “no-hair theorem,” once the charge is captured by the black hole, the angular momentum associated with the charge monopole system loses all traces of its exotic origin and is perceived from the outside as common rotation. It is argued that a similar analysis performed on Taub–NUT space should give the same result.

Taub–NUT space

Magnetic poles and black holes are remarkable objects. To some extent they have had similar histories.

The black hole emerged as a solution of the Einstein equations that was at first regarded as unphysical because of its singular nature. However, further study for many years by many researchers demonstrated that black holes were physically relevant as the endpoint of gravitational collapse (see, e.g., ref. 1). But, even then, collapsed objects were thought to be scarce, and the observational search for them began. Nowadays it is commonly accepted that enormous black holes exist at the centers of galaxies, including our own (2). Furthermore, those black holes may be actually responsible for the very existence of the galaxies themselves and, therefore, for the presence of structure in the universe.

It is no minor feat that in <70 years (3) the black hole has risen from the status of an unphysical exotic solution of the Einstein equations to being an observed astrophysical object responsible for structure in the universe.

The magnetic pole was first introduced as an appealing modification of Maxwell’s equations, which, without it, are not fully symmetric under duality rotations of the electric and magnetic fields. (4, 5, 1). The introduction of the magnetic pole had a spectacular implication, namely, that the mere existence of a single magnetic pole in the universe would imply that all electric charges should be integer multiples of a basic quantum that is inversely proportional to the magnetic charge of the pole. This provided the first possible theoretical basis for a hitherto totally unexplained key feature of the universe, the quantization of electric charge.

However, the introduction of magnetic poles was a modification that, although attractive for aesthetic reasons, was not implied by Maxwell’s equations themselves, which had been amply validated by experiment. But then help came from a different avenue of inquiry, namely the attempt to find a theory

that incorporated the charge independence of nuclear forces, which gave rise to the Yang–Mills fields. Magnetic poles were shown to arise as solutions of the field equations of an $SU(2)$ Yang–Mills theory coupled to a scalar field multiplet. The solution was regular, and the electric and magnetic charges obeyed the Dirac quantization condition (6, 7). At that point, the focus of Yang–Mills theory had already shifted from the charge independence of the nuclear forces (gauge theory of isospin) to the interaction of subnuclear matter, and grand unified theories. These grand unified theories predict the existence of monopoles (see, e.g., chapter 23 of ref. 8).

So it would appear well founded to say that the magnetic pole has gained full theoretical respectability and that it is now not an option but a consequence of accepted microphysical theory. However, this success story is not at the same level as that of the black hole. Indeed, magnetic poles have not been observed, and, more importantly, we lack a distinct fundamental role for them in our present view of the universe.

Nevertheless, if the history of the black hole is to teach us a lesson, it is that it might not be totally out of the question to think that this simple fundamental object has not yet found its proper central place in physics but it should at some point do so.

With the above motivation in mind, we have undertaken the study of the simplest problem where these two remarkable objects, the black hole and the magnetic pole, interact.

For this study, we first recall (in *Monopole Angular Momentum Revisited*) the striking property of an electric charge near a magnetic pole in flat space, which is that the system possesses angular momentum even when both the electric and the magnetic charges are at rest. The angular momentum is proportional to the product of the charges. In this process we develop an economic way to compute the angular momentum, which also clarifies possible confusion about Dirac strings and the like.

Next, in *Electric Charge Near a Magnetic Black Hole*, we study an electric charge held at rest outside a magnetically charged black hole. This situation is equivalent, by electromagnetic duality, to the case of a magnetic pole placed at rest in the background of an electric black hole announced in our title. We find that, even if the electric charge is treated as a perturbation on a spherically symmetric magnetic Reissner–Nordstrom hole, the geometry at large distances is that of a magnetic Kerr–Newman black hole. When the charge approaches the horizon and crosses it, the exterior geometry becomes that of a Kerr–Newman hole with electric and magnetic charges and with total angular momentum given by the standard value for a charged monopole pair. Thus, in accordance with the “no-hair theorem,” once the charge is captured by the black hole, the angular momentum associated with the charge monopole system loses all

Author contributions: C.B. and M.H. performed research and wrote the paper.

The authors declare no conflict of interest.

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[‡]See also Wheeler, J. A., Eleventh Solvay Conference on Physics, June 9–13, 1958, Brussels, Belgium.

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traces of its exotic origin and it is perceived from the outside as common rotation.

Higher Spin Poles is devoted to arguing that a similar analysis performed on Taub–NUT space would give the same result, namely, if one holds an ordinary mass outside the horizon of a Taub–NUT space with only magnetic mass, the system as seen from large distances is endowed with an angular momentum proportional to the product of the two kinds of masses. When the ordinary electric mass reaches the horizon, the exterior metric becomes that of a rotating Taub–NUT space. This rotating space, “Kerr–Taub–NUT geometry,” is a solution of the vacuum Einstein equations different from ordinary Taub–NUT space (Taub–NUT space does not possess angular momentum in spite of having both electric and magnetic mass).

Finally, *Conclusions* is devoted to brief concluding remarks. Among them we observe that through successive captures of electric and magnetic poles of both signs, a Schwarzschild black hole can become a (neutral) Kerr hole. We then indulge in the speculation that monopoles might account for some of the rotation of the black holes in the universe.

Monopole Angular Momentum Revisited

We consider a magnetic pole of strength g at the origin of coordinates in flat space and an electric charge q located at a distance c above the magnetic pole on the z axis. The magnetic and electric fields are

$$\vec{B} = \frac{g\vec{r}}{r^3}, \quad \vec{E} = \frac{q(\vec{r} - \vec{c})}{|\vec{r} - \vec{c}|^3}, \quad [1]$$

with $c = c\hat{e}_3$. They obey the Gauss law equations

$$\vec{\nabla} \cdot \vec{B} = 4\pi g \delta^{(3)}(\vec{r}) \text{ and} \quad [2]$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi q \delta^{(3)}(\vec{r} - \vec{c}).$$

We will use the standard “electric picture” and introduce a vector potential for the magnetic field, which has only a nonvanishing azimuthal component,

$$A = g(k - \cos\theta)d\phi, \quad [3]$$

as well as the magnetic and electric field densities

$$\mathcal{B}^i = \sqrt{g} B^i, \quad \mathcal{E}^i = \sqrt{g} E^i. \quad [4]$$

The formula

$$\mathcal{B}^i = \varepsilon^{ijk} \partial_j A_k \quad [5]$$

reproduces the field given in Eq. 1. However, the potential (Eq. 3) is well defined only away from the z axis for a general value of k . The singularity of the potential on the z axis may be pictured in physical terms as a concentrated flux coming in along the positive and negative z axes with strengths that add up to g . This flux reemerges then radially from the origin to give, away from the z axis, the field (Eq. 1). To compensate for this singular flux, one normally brings in an additional entity, the Dirac string, which cancels the flux and therefore the right side of Eq. 5 acquires an additional contribution in order to be valid also on the z axis. It is important to emphasize that the Dirac string is not the singularity of A_ϕ but, rather, the additional object, which is brought in to cancel it. As a consequence, if one is only interested in the B field given in Eq. 1, as we will be in the present paper, one may just take Eq. 5 as is and simply extrapolate continuously its value to the z axis.

Thus, in computing the angular momentum stored in the field

$$\vec{J} = -\frac{1}{4\pi} \int_V \vec{r} \times (\vec{E} \times \vec{B}) d^3x \quad [6]$$

we may substitute Eq. 5 for B and obtain the correct answer, as we shall proceed to do.

For symmetry reasons, the only nonvanishing component of the angular momentum \vec{J} will be along the z axis. Customarily, one tackles the integral directly in Cartesian coordinates. However, it will be simpler, and useful to us further below, to work in spherical coordinates, recalling that the z component of the angular momentum, is simply the azimuthal component of the linear momentum whose density is the Poynting vector

$$-\frac{1}{4\pi} F_{\phi j} \varepsilon^j. \quad [7]$$

Therefore, the only nonvanishing component of Eq. 6 reads

$$J_z = -\frac{1}{4\pi} \int dr d\theta d\phi F_{\phi j} \varepsilon^j. \quad [8]$$

We evaluate the integral as follows. First, we note that B and A_ϕ depend only on θ , while ε depends only on r and θ . Furthermore, from Eq. 1, the only vanishing components of ε^i are ε^r and ε^θ , so we can rewrite Eq. 8 as,

$$J_z = \frac{1}{4\pi} \int dr d\theta d\phi \partial_\theta A_\phi \varepsilon^\theta. \quad [9]$$

Now we observe from Eq. 1 that ε^θ vanishes at $\theta = 0$ and $\theta = \pi$. Therefore, after integration by parts in θ , we obtain

$$J_z = -\frac{1}{4\pi} \int dr d\theta d\phi A_\phi \partial_\theta \varepsilon^\theta. \quad [10]$$

Next we rewrite Eq. 10 by introducing the explicit value (Eq. 3) for A_ϕ and repeatedly using Gauss’s law for the electric field, which, written in spherical coordinates, reads

$$\partial_r \varepsilon^r + \partial_\theta \varepsilon^\theta = 4\pi q \delta(r - c) \delta^{(2)}(\theta, \phi), \quad [11]$$

where $\delta^{(2)}(\theta, \phi)$ is the δ function density defined on the sphere with support at the northern pole, that is $\int f(\theta, \phi) \delta^{(2)}(\theta, \phi) = f(\theta = 0)$. We obtain for the angular momentum contained in a region of space bounded by the two spheres of radii r_1 and r_2

$$J_z = \int_{r_1}^{r_2} dr j(r), \quad [12]$$

with

$$j = \frac{d}{dr} \left[gqH(r - c) - \frac{g}{2} \int_0^\pi d\theta \cos \theta \varepsilon^r(r, \theta) \right], \quad [13]$$

where $H(r - c)$ is the Heaviside step function ($H = 0$ for $r < c$, $H = 1$ for $r > c$). Note that the constant k drops out of the final answer, as it should since the magnetic field does not depend on it.

Eq. 13 for the effective radial density $j(r)$ of J_z has the remarkable property of being the derivative (“divergence”) of a local function of r , which means that one can write the angular momentum contained between the two spheres as the difference between two “surface integrals” (“fluxes”), namely,

$$J_z = \Phi(r_2) - \Phi(r_1), \quad [14]$$

with

$$\Phi(r) = gqH(r - c) - \frac{g}{2} \int_0^\pi d\theta \cos \theta \varepsilon^r(r, \theta). \quad [15]$$

Note in particular that

$$\Phi(0) = 0. \quad [16]$$

Eq. 16 follows from (i) the Heaviside function vanishes because $c > 0$, (ii) near the origin, $\varepsilon^r = qr^2/c^2 \sin \theta \cos \theta$, which vanishes at $r = 0$. It states that there is no δ -function source for the angular momentum on the magnetic pole. By symmetry, there is no δ -function source for the angular momentum at the electric pole either.

As a consequence of Eqs. 14 and 16, the angular momentum within a sphere of radius r is

$$J_z = \Phi(r), \quad [17]$$

and, for the total angular momentum, we recover the standard result

$$J_z = \Phi(\infty) = qg. \quad [18]$$

This follows from (i) the Heaviside function is equal to one because $c < \infty$, (ii) near infinity, $\varepsilon^r = q \sin \theta$, and this expression, once multiplied by $\cos \theta$ and integrated over θ , yields zero. As it is well known, if one demands that J_z given by Eq. 18 should be quantized to half-integer values, one obtains the Dirac quantization condition

$$\frac{2qg}{\hbar} \in \mathbb{Z}. \quad [19]$$

Electric Charge Near a Magnetic Black Hole

We now consider the case of an electric charge q placed at rest in the background of a black hole with magnetic charge g . Just as before, we place the electric charge on the positive z axis at $r = c$. This situation is equivalent, by electromagnetic duality, to the case of a magnetic pole placed at rest in the background of an electric black hole and permits direct contact with the preceding discussion in flat space.

The unperturbed geometry is described by the Reissner-Nordstrom metric,

$$ds^2 = - \left(1 - \frac{2M}{r} + \frac{g^2}{r^2} \right) dt^2 + \left(1 - \frac{2M}{r} + \frac{g^2}{r^2} \right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad [20]$$

and the unperturbed electromagnetic field is purely magnetic and given by the expression

$$F = g \sin \theta d\theta \wedge d\phi, \quad [21]$$

just as in the flat space case.

We first determine, to first order in q , the components of the perturbed fields relevant to the computation of the angular momentum. To that effect, we observe that the perturbation of the electromagnetic field is purely electric, and that, just as before, the electric field ε^i , whose divergence gives a δ -function at the location of the charge, has no azimuthal component. Furthermore, ε^θ and ε^r depend only on r and θ .

In general relativity, the total angular momentum is given by a surface integral at infinity because it is a global conserved charge associated with a gauged symmetry. The surface integral is determined by the requirement that the corresponding generator should have well defined functional derivatives (9). In the

case at hand (rotations about the z axis), the generator can be taken to be

$$G = \int d^3x \xi^\phi \mathcal{H}_\phi + J_z. \quad [22]$$

Here ξ^ϕ is an arbitrary function of the spatial coordinates (“surface deformations” and “gauged rotations”), which tends to unity at infinity.

$$\xi^\phi \rightarrow 1, \quad r \rightarrow \infty. \quad [23]$$

The Hamiltonian generator \mathcal{H}_ϕ is given by

$$\mathcal{H}_\phi = -2\pi^k_{\phi|k} - \frac{1}{4\pi} F_{\phi k} \varepsilon^k. \quad [24]$$

Here, π^{ij} is the canonical conjugate to the spatial metric g_{ij} , ε^k is the electric field density, which is proportional to the canonical conjugate π^k of the potential A_k ($\pi^k = (1/4\pi)\varepsilon^k$), and the vertical bar denotes covariant differentiation in g_{ij} .

The surface integral J_z is determined by the demand that the variation of the generator G should be given by the volume integral of a local function containing no derivatives of the variations of the dynamical variables. Thus, in practice J_z is constructed to compensate for the surface integrals at infinity, which arise upon integration by parts in the volume piece of δG . In order to implement this procedure, it is necessary to give boundary conditions at infinity for all the fields. These boundary conditions include definite parity conditions (behavior under $\theta \rightarrow \pi - \theta$, $\phi \rightarrow \phi + \pi$) (9). In particular, the vector potential should be odd to leading order. This means that in the Hamiltonian treatment one must take the arbitrary constant k appearing in Eq. 3 equal to zero. Therefore, the vector potential for the field strength (Eq. 21) will be taken to be

$$A = -g \cos \theta d\phi. \quad [25]$$

To determine J_z in the case at hand, it is sufficient to write the Hamiltonian generator \mathcal{H}_ϕ , taking g_{ij} to be the spatial metric of the background (Eq. 20) and allowing for a perturbation $\pi^{ij}(r, \theta)$ of the background (Eq. 20), which has zero π^{ij} . To begin the analysis, we write \mathcal{H}_ϕ explicitly. The calculation is quite simple because the symmetrized combinations of the Christoffel symbols that appear to vanish due to the simple form of the Reissner-Nordstrom metric. One finds

$$\mathcal{H}_\phi = -2\pi^k_{\phi, k} - \frac{1}{4\pi} F_{\phi k} \varepsilon^k. \quad [26]$$

The variation of G then gives

$$\delta G = \text{Volume integral} - 2 \int_{S^2_\infty} \delta \pi^r_\phi d\theta d\phi, \quad [27]$$

from which we conclude that

$$J_z = 2 \int_{S^2_\infty} \pi^r_\phi d\theta d\phi. \quad [28]$$

In order to evaluate the surface integral (Eq. 28), we first integrate the constraint equation

$$\mathcal{H}_\phi = 0 \quad [29]$$

over the two-sphere and from an arbitrary fixed value of r to infinity. For the electromagnetic contribution, we can take over the results of the previous section and we obtain

$$J_z = \Phi(\infty) - \Phi(r_+) + 2 \int_{S^2(r_+)} \pi_\phi^r d\theta d\phi. \quad [30]$$

Now, we observe that due to the conservation of angular momentum, J_z should be independent of the radial coordinate c of the electric charge. Indeed, one can imagine displacing the electric charge in the radial direction from one location to another. This could be accomplished, for example, by letting it fall and then stopping it, or, say, by moving it adiabatically by holding it with a rope. In either case, the force exerted on the charge will be radial and therefore would exert no torque around the origin. This reasoning also applies in flat space and it is in that case yet another way, besides dimensional analysis, to realize that the angular momentum of the electric-magnetic pole pair is independent of the distance between the monopoles.

It is thus sufficient to evaluate the integrals in Eq. 30 for $c \gg r_+$. This, of course, is the same as letting $r_+ \rightarrow 0$ keeping c finite. The space then becomes flat, and the domain of integration for each of the two integrals at $r = r_+$ becomes a two-sphere of vanishing radius, which makes each integral vanish because the integrand is regular. Here, we are using the word “regular” in the geometrical sense. This means that a regular density of positive weight vanishes at the origin when expressed in polar coordinates. So, we find again

$$J_z = \Phi(\infty) = qg \quad [31]$$

(for any c).

We may think of Eq. 30 as stating that the total angular momentum is composed of two parts, the angular momentum stored in the electromagnetic field

$$\Phi(\infty) - \Phi(r_+), \quad [32]$$

and the spin of the black hole,

$$2 \int_{S^2(r_+)} \pi_\phi^r d\theta d\phi. \quad [33]$$

Imagine now that the point charge is moved toward the horizon and crosses it. Once the charge is inside the horizon, we are faced with the Einstein–Maxwell equations with both electric and magnetic charges with two Killing vectors $\partial/\partial t$, $\partial/\partial \phi$. By the black hole uniqueness theorem, the exterior solution is then the Kerr–Newman metric (1, 10, 11) with electric and magnetic charges with a corresponding electromagnetic field, linearized in the electric charge and with a value for the total angular momentum given by Eq. 31. That line element will be explicitly displayed in Eq. 37.

What happens is that when the charge is far away from the horizon, one has a nonrotating black hole, and the angular momentum is all stored in the electromagnetic field outside the horizon. As the charge is brought in, the angular momentum in the field starts being continuously transferred to the hole, which begins to spin around faster and faster as the charge gets closer and closer to r_+ . Thus, Eq. 32 decreases in magnitude from qg to the value that it has for Kerr–Newman (with both q and g). This value is not zero, as it would be for Reissner–Nordstrom, since the rotation there is a non-zero component ε^θ . At the same time, Eq. 33 increases from zero to

$$2 \int_{S^2(r_+)} \pi_\phi^r d\theta d\phi = -Ma + \frac{2}{3} \frac{g^2 a}{r_+}, \quad [34]$$

where

$$-Ma = qg. \quad [35]$$

The spin of the hole (Eq. 34) differs from the total angular momentum by the residual angular momentum in the Kerr–Newman electromagnetic field

$$\Phi(\infty) - \Phi(r_+) = -\frac{2}{3} \frac{g^2 a}{r_+}. \quad [36]$$

In order not to interrupt the thread of the argument, the derivation of Eqs. 34 and 36 is given in *Appendix 1*.

When the charge reaches the horizon, the transfer has become complete and the black hole is rotating exactly at the required rate so that the charge can go in smoothly without giving the hole a jolt. It is as if a child wants to get on a merry-go-round without hitting himself when he jumps on it. He must then run so that he reaches the platform with the same angular velocity as the merry-go-round. The trick here is of course that gravity does the job for the child by adjusting the angular velocity of the merry-go-round so that the child can approach in any way he wishes (even radially).

It is important to realize how different the situation is for the black hole case from the flat space case described in the previous section. In flat space, when the electric charge approaches the magnetic pole, the electromagnetic angular momentum density is changed and tends to pile up near the origin. However, the integral of that density is unchanged, and, therefore, the total angular momentum is not transferred from the electromagnetic field to anything else. The pair (q, g) does not start spinning around as the charges get closer. It just stays at rest. On the other hand, in the black hole case, the hole acquires an intrinsic spin which leaves the same imprint on the geometry as the one that would occur if the hole had been formed by the collapse of a rotating star. The “transfer” only exists in the presence of the gravitational coupling, which provides the mechanism for its occurrence and which also is responsible for the existence of the black hole to begin with.

To end this section, it should be made clear that, for any location of the electric charge, one may go to radial distances well beyond it towards infinity. At those large distances, the metric coincides with the asymptotic form of the Kerr–Newman line element, which reads explicitly (when linearized in q)

$$\begin{aligned} ds^2 = & -\frac{\Delta}{r^2} dt^2 + r^2 \sin^2 \theta d\phi^2 \\ & + \frac{2a \sin^2 \theta}{r^2} [-2Mr + g^2] dt d\phi \\ & + \frac{r^2}{\Delta} dr^2 + r^2 d\theta^2, \end{aligned} \quad [37]$$

with

$$\Delta = r^2 - 2Mr + g^2. \quad [38]$$

As the charge moves in and the “hair” progressively disappears, the approximation of the actual metric by the Kerr–Newman line element becomes more and more accurate for all distances until it is exact when the charge reaches the horizon.

Higher Spin Poles

It has been shown recently that one may extend the notion of a magnetic pole to higher integer spin gauge fields (11). For spin s , the corresponding “electric” and “magnetic” conserved charges are symmetric tensors of rank $(s - 1)$ $P_{v_1 v_2 \dots v_{s-1}}$ and $Q_{v_1 v_2 \dots v_{s-1}}$, and the analog of the Dirac quantization condition is

$$\frac{1}{2\pi\hbar} Q_{v_1 v_2 \dots v_{s-1}} P^{v_1 v_2 \dots v_{s-1}} \in \mathbb{Z}. \quad [39]$$

The case previously considered has dealt with black holes that may possess both electric and magnetic $s = 1$ charges, and the underlying theory is the Einstein–Maxwell theory. It would be natural to attempt an extension of the discussion to higher spins. However, with our present stage of knowledge, this is only possible for $s = 2$. This is because we do not know how to couple gravity to higher spin fields and sources.

For $s = 2$, the theory exists, and it is just the Einstein theory, where we know the analog of a black hole with both electric and magnetic sources, which is the Taub–NUT space (12, 13). We also know that an ordinary (“electric”) test mass moves along a geodesic in a Taub–NUT field.

With these two elements, the total angular momentum of the system formed by a test electric mass at $r = c$ on the z axis of a magnetic Taub–NUT background was evaluated in ref. 11. If, following the customary notation that will be made explicit below in Eq. 43, the magnetic mass is denoted by N and the test electric mass by m , the angular momentum is given by

$$J_z = 2Nm, \quad [40]$$

and it is again independent of the separation of the electric and magnetic masses.

Knowing Eq. 40 and using the insight provided by the previous analysis, we will limit ourselves to describing, without making any attempt to prove, what one expects to happen as the mass is lowered onto the magnetic hole from a large distance.

When the mass is very far, the black hole is not rotating, and, at distances $r_+ \leq r \gg c$, the line element is that of a purely magnetic Taub–NUT space, namely

$$ds^2 = -V(r)[dt - 2N \cos \theta d\phi]^2 + V(r)^{-1} dr^2 + (r^2 + N^2)(d\theta^2 + \sin^2 \theta d\phi^2), \quad [41]$$

with

$$V(r) = 1 - \frac{2N^2}{(r^2 + N^2)} = \frac{r^2 - N^2}{r^2 + N^2}. \quad [42]$$

At distances well beyond the electric mass, $r \gg c$, the metric will asymptotically coincide with the leading approximation for large r of a Kerr–Taub–NUT space (14) with electric mass m , magnetic mass N , and angular momentum $J_z = -ma = 2Nm$,

$$ds^2 = -\frac{1}{\Sigma} (\Delta - a^2 \sin^2 \theta) dt^2 + \frac{2}{\Sigma} [\Delta \chi - a(\Sigma + a\chi) \sin^2 \theta] dt d\phi + \frac{1}{\Sigma} [(\Sigma + a\chi)^2 \sin^2 \theta - \chi^2 \Delta] d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2. \quad [43]$$

Here, we have set

$$\Sigma = r^2 + (N - a \cos \theta)^2 \quad [44]$$

$$\Delta = r^2 - 2mr - N^2 + a^2 \quad [45]$$

$$\chi = a \sin^2 \theta + 2N \cos \theta. \quad [46]$$

As the mass gets lowered, the outside Kerr–Taub–NUT approximation gets better and better until it becomes exact when the mass reaches the horizon.

In this case, just as in the electromagnetic case, one could say that one got the black hole to turn more and more as the electric mass approaches it. This time, however, the “transfer” of angular momentum has not been from the electromagnetic field of the source to the black hole, but rather, from the gravitational field of the perturbation (which can be unambiguously separated from the background) to the gravitational field of the hole.

Conclusions

We have analyzed how the presence of an electric charge in its exterior perturbs a magnetically charged, nonrotating black hole. Because of the invariance of the equations under electromagnetic duality, this situation is equivalent to placing a magnetic pole outside an electrically charged black hole. At large distances, the geometry is that of a magnetic Kerr–Newman black hole. When the charge approaches the horizon and crosses it, the “hair” is lost and the exterior geometry becomes exactly that of a Kerr–Newman hole with electric and magnetic charges and with total angular momentum given by the standard value for a charged monopole pair. Thus, in accordance with the “no-hair theorem,” once the charge is captured by the black hole, the angular momentum associated with the charge monopole system loses all traces of its exotic origin and is perceived from the outside as common rotation.

We have argued that a similar analysis performed on Taub–NUT space should give the same result; namely, if one holds an ordinary mass outside of the horizon of a Taub–NUT space with only magnetic mass, the system, as seen from large distances, is endowed with an angular momentum proportional to the product of the two kinds of masses. When the ordinary electric mass reaches the horizon, the exterior metric becomes that of a rotating Taub–NUT space. This rotating space (Kerr–Taub–NUT metric) is a solution of the vacuum Einstein equations different from ordinary Taub–NUT space, which, in spite of having both electric and magnetic mass, does not possess angular momentum.

It is quite remarkable that one may set a black hole in rotation by radially throwing into it a magnetic pole. One may even obtain, through successive applications of this process, a rotating black hole that is neutral both electrically and magnetically. Indeed, suppose that one starts with a Schwarzschild hole and consider the following chain of four successive processes:

1. Radially throw in a charge $+q$ through the northern pole. One then gets a Reissner–Nordstrom black hole (electrically charged, nonrotating).
2. Now, radially throw in a magnetic charge $+g$ through the northern pole. One then gets a Kerr–Newman black hole (electric charge q , magnetic charge g , angular momentum $J_z = -gq/4\pi$).
3. Next, radially throw in through the southern pole a magnetic charge $-g$. One then gets a Kerr–Newman black hole with no magnetic charge but rotating twice as fast (electric charge q , magnetic charge zero, angular momentum $J_z = -gq/2\pi$).
4. Finally, again radially throw in through the southern pole an electric charge $-q$. One ends up with a Kerr black hole with vanishing total electric charge, vanishing total magnetic charge and angular momentum $J_z = -gq/2\pi$.

After this sequence of processes is completed, it is impossible to tell that the Kerr black hole that has been formed had anything to do with electric or magnetic monopoles. However, their existence was necessary to set the black hole in rotation in this manner.

It is perhaps not totally inconceivable to imagine that our universe is such that, at least part of the rotation of some of the black holes that we have observed, might come from their hiding the magnetic poles that we have not yet observed.

Appendix 1: Establishing Eqs. 34 and 36

Evaluation of Eq. 34. To compute the integral (Eq. 34), we first evaluate the extrinsic curvature component

$$K_{\phi r} = \frac{1}{2N} (N_{\phi|r} + N_{r|\phi}) \quad [47]$$

of the slices on which t is constant of the metric of Eq. 37. One has

$$g_{\phi r} = -\frac{2Ma}{r} \sin^2 \theta + \frac{g^2 a}{r^2} \sin^2 \theta, \quad [48]$$

and hence

$$N_{\phi|r} + N_{r|\phi} = \frac{6Ma}{r^2} \sin^2 \theta - \frac{4g^2 a}{r^3} \sin^2 \theta. \quad [49]$$

The lapse is equal to

$$N = \left(1 - \frac{2M}{r} + \frac{g^2}{r^2} \right)^{1/2}$$

(to first order in q), and hence

$$K_{\phi r} = \frac{\frac{6Ma}{r^2} \sin^2 \theta - \frac{4g^2 a}{r^3} \sin^2 \theta}{2 \left(1 - \frac{2M}{r} + \frac{g^2}{r^2} \right)^{1/2}}. \quad [50]$$

The momentum component π_{ϕ}^r is related to $K_{\phi r}$ by

$$\pi_{\phi}^r = -\frac{1}{16\pi} K_{\phi r} g^{rr} \sqrt{g}, \quad [51]$$

which leads to

$$-32\pi \pi_{\phi}^r = 6M a \sin^3 \theta - \frac{4g^2 a}{r} \sin^3 \theta. \quad [52]$$

Integration over the angles gives then the announced result at $r = r_+$,

$$2 \int_{S(r_+)} \pi_{\phi}^r d\theta d\phi = -Ma + \frac{2g^2 a}{3r_+}. \quad [53]$$

Direct Evaluation of Eq. 36. We evaluate directly the difference $\Phi(\infty) - \Phi(r_+)$ when the electric charge is near the horizon, $c = r_+ + \varepsilon$. To that end, we observe that $\Phi(r)$ is a continuous function at $r = c$ (the discontinuity in the Heaviside function is compensated by the same discontinuity in the integral of ε^r). This statement was proven for flat space in *Monopole Angular Momentum Revisited* (recall the discussion following Eq. 16). This continuity stays valid when the test charge is placed on the curved background of the magnetic pole because, at the location of the electric charge, that background can be obtained by a smooth deformation of flat space. Therefore, we can compute equivalently $\Phi(\infty) - \Phi(r_+)$ when the electric charge

has just plunged into the black hole. In that case, $\Phi(\infty) - \Phi(r_+)$ reduces to

$$\Phi(\infty) - \Phi(r_+) = \frac{g}{2} \int_0^\pi d\theta \cos \theta \varepsilon^r(r_+, \theta), \quad [54]$$

where the electric field is that of a Kerr–Newman black hole with electric charge q , magnetic charge g , and angular momentum qg .

The radial component ε^r of the electric field at r_+ has two pieces: ε_1^r , which comes from the electric charge, and ε_2^r , which comes from the rotation of the magnetic hole. To first order in q , they are given by

$$\varepsilon_1^r = q \sin \theta \quad [55]$$

$$\varepsilon_2^r = -\frac{2ga}{r_+} \cos \theta \sin \theta. \quad [56]$$

Only ε_2^r contributes to the integral of Eq. 54.

$$\begin{aligned} \Phi(\infty) - \Phi(r_+) &= -\frac{g^2 a}{r_+} \int_0^\pi d\theta \cos^2 \theta \sin \theta \\ &= -\frac{2}{3} \frac{g^2 a}{r_+}, \end{aligned} \quad [57]$$

which is the announced formula (Eq. 36).

Strictly speaking, once Eq. 34 is proven, it is not necessary to establish Eq. 36 directly because it follows from Eqs. 30 and 31 that

$$\Phi(\infty) - \Phi(r_+) = J_z - 2 \int_{S(r_+)} \pi_{\phi}^r d\theta d\phi \quad [58]$$

and $J_z = -Ma$. Nevertheless, we have included this derivation because we believe that it provides additional insight on the mechanism through which the capture of a magnetic pole sets a black hole in rotation.

Note. After this work was submitted, we learned of an interesting paper (15) in which the angular momentum of an electric charge at rest in the field of a magnetic black hole was computed. Our work is complementary to this insightful article in that we consider in more detail the dynamical transfer of angular momentum to the magnetic black hole as the electric charge falls. We take a boundary condition different from the one imposed in ref. 15; namely, we demand that the hole is nonrotating when the charge is very far from it. On the other hand, in ref. 15, it is imposed that the spin of the hole is zero when the charge is at a given distance b . This means that when the charge is at infinity, the hole is rotating in such a way that the sum of its initial angular momentum and the angular momentum it receives when the charge gets from infinity to b exactly vanishes. For this reason, the total angular momentum depends on b in their case, while it does not in ours. We also check the disappearance of the associated hair when the black hole forms by verifying the match of the relevant surface integrals on the horizon of the Kerr–Newman black hole before and after the electric charge has plunged in. We thank the authors of ref. 15 for calling our attention to their work. Another paper that was brought to our attention after this work was submitted is ref. 16. These authors discuss the angular momentum of a test magnetic charge and a test electric charge in a curved, axisymmetric (horizon-free) background spacetime and verify that it is equal to the product qg , independently of the curvature. This is, in fact, a direct consequence of our Eq. 30 with $r_+ = 0$.

We thank the skipper and crew of the schooner *Raquel* for generous hospitality and support and Rubén Portugues for key discussions in the early stages of this research. This work was supported by an institutional

grant from the Millennium Science Initiative, Chile (to Centro de Estudios Científicos), and from the generous support of Empresas CMPC (to Centro de Estudios Científicos). M.H. was supported in part by Institut Interuniversitaire des Sciences Nucléaires, Belgium (Con-

vention 4.4505.86), the Interuniversity Attraction Poles Program–Belgian Science Policy, and by European Commission Program MRTN-CT-2004–005104 (in which M.H. is associated to Vrije Universiteit Bruxelles).

1. Misner CW, Thorne KS, Wheeler JA (1973) *Gravitation* (Freeman, San Francisco).
2. Alexander T (2005) *Phys Rep* 419:65.
3. Oppenheimer JR, Snyder H (1939) *Phys Rev* 56:455.
4. Dirac PAM (1931) *Proc R Soc London Ser A* 133:60.
5. Dirac PAM (1948) *Phys Rev* 74:817.
6. Hooft G (1974) *Nucl Phys B* 79:276.
7. Polyakov AM (1974) *JETP Lett* 20:194.
8. Weinberg S (1996) *The Quantum Theory of Fields* (Cambridge Univ Press, Cambridge, UK), Vol 2.
9. Regge T, Teitelboim C (1974) *Ann Phys (Oxford, UK)* 88:286.
10. Kerr RP (1963) *Phys Rev Lett* 11:237.
11. Newman ET, Couch R, Chinnapared K, Exton A, Prakash A, Torrence R (1965) *J Math Phys* 6:918.
12. Bunster C, Cnockaert S, Henneaux M, Portugues R (2006) *Phys Rev D* 73:105014.
13. Newman E, Tamubirino L, Unti T (1963) *J Math Phys* 4:915.
14. Misner CW (1963) *J Math Phys* 4:924.
15. Demiański M, Newman ET (1966) *Bull Acad Pol Sci* 14:653.
16. Garfinkle D, Rey SJ (1991) *Phys Lett B* 257:158.
17. Friedman JL, Mayer S, Parker L (1978) *Phys Rev D* 17:1957.