Perspectives of matrix convex functions

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In this paper, we generalize the main results of [Effros EG, (2009) *Proc Natl Acad. Sci USA* 106:1006–1008]. Namely, we provide the necessary and sufficient conditions for jointly convexity of perspective functions and generalized perspective functions.

matrix convex function

We denote by $H_n(\mathbb{I})$ and $G_n(\mathbb{I})$, the self-adjoint $n \times n$ matrices with spectra in the closed interval $\mathbb{I}$ and the strictly positive $n \times n$ matrices with spectra in $\mathbb{I}$, respectively. Throughout this paper, it is assumed that f is a real-valued continuous function on $\mathbb{I}$. We say that f is matrix (operator) convex if

$$f(cL_1 + (1-c)L_2) \le cf(L_1) + (1-c)f(L_2)$$

for all $L_1,L_2 \in H_n(\mathbb{I})$ and all $c \in [0,1]$. Also, f is matrix concave if $-f$ is matrix convex. Given $L \in H_n(\mathbb{I})$, the value $f(L)$ is defined by the spectral theorem, i.e., if $L = \sum_{i=1}^m \lambda_i P_i$ is the spectral decomposition of L, then $f(L) =: \sum_{i=1}^m f(\lambda_i) P_i$. We define the perspective function g (associated to f) on $H_n(\mathbb{I}) \times G_n(\mathbb{I})$ by

$$(L,R) \longmapsto g(L,R) =: R^{1/2} f(R^{-1/2} L R^{-1/2}) R^{1/2}.$$

Let g be a perspective function. We say that g is jointly convex if

$$\begin{aligned} g(cL_1 + (1-c)L_2, cR_1 + (1-c)R_2) \\ \le cg(L_1,R_1) + (1-c)g(L_2,R_2) \end{aligned}$$

for all $L_1,L_2 \in H_n(\mathbb{I})$, $R_1,R_2 \in G_n(\mathbb{I})$, and all $c \in [0,1]$. Also, g is jointly concave if $-g$ is jointly convex.

Let $h > 0$ be a continuous function. We define the generalized perspective function (associated to f and h) on $H_n(\mathbb{I}) \times G_n(\mathbb{I})$ with

$$(L,R) \longmapsto (f \triangle h)(L,R) =: h(R)^{1/2} f(h(R)^{-1/2} L h(R)^{-1/2}) h(R)^{1/2}.$$

Recall that if for any continuous function f, $f(L)$ commutes with any operator commuting with L (including L itself), then we obtain the perspective function $g(L,R) = f(\frac{L}{R})R$ which is considered in ref. 1 by Edward G. Effros.

In this note, we provide the necessary and sufficient conditions between matrix convexity and jointly convexity of perspective and generalized perspective functions.

Main Results

Recently, Edward G. Effros (1) proved the following theorem.

Theorem 2.1. *Suppose that f is operator convex. When restricted to positive commuting matrices L,R, the perspective function*

$$(L,R) \mapsto g(L,R) = f\left(\frac{L}{R}\right) R$$

is jointly convex in the sense that if $L = cL_1 + (1-c)L_2$ and $R = cR_1 + (1-c)R_2$ where $[L_j,R_j] = 0$ $(j = 1,2)$, and $0 \le c \le 1$, then

$$g(L,R) \le cg(L_1,R_1) + (1-c)g(L_2,R_2).$$

In the following, we remove the conditions $[L,R] = 0$, and $[L_j,R_j] = 0$ $(j = 1,2)$ in the above theorem. Moreover, we prove that the jointly convexity of g implies the matrix convexity of f. We use Theorem 2.1 of ref. 2 to prove the main theorem of our paper.

Theorem 2.2. *The function f is matrix convex if and only if the perspective function g is jointly convex.*

Proof: Let f be matrix convex and let $L_1,L_2 \in H_n(\mathbb{I})$, $R_1,R_2 \in G_n(\mathbb{I})$, and $0 \le c \le 1$. Put $L =: cL_1 + (1-c)L_2$ and $R =: cR_1 + (1-c)R_2$. The matrices $T_1 =:= (cR_1)^{1/2} R^{-1/2}$ and $T_2 =: ((1-c)R_2)^{1/2} R^{-1/2}$ satisfy $T_1^* T_1 + T_2^* T_2 = 1$. From matrix convexity of f and Theorem 2.1 of ref. 2, we have

$$\begin{aligned} g(L,R) &= R^{1/2} f(R^{-1/2} L R^{-1/2}) R^{1/2} \\ &= R^{1/2} f(T_1^* R_1^{-1/2} L_1 R_1^{-1/2} T_1 + T_2^* R_2^{-1/2} L_2 R_2^{-1/2} T_2) R^{1/2} \\ &\le R^{1/2} (T_1^* f(R_1^{-1/2} L_1 R_1^{-1/2}) T_1 \\ &\quad + T_2^* f(R_2^{-1/2} L_2 R_2^{-1/2}) T_2) R^{1/2} \\ &= c R_1^{1/2} f(R_1^{-1/2} L_1 R_1^{-1/2}) R_1^{1/2} \\ &\quad + (1-c) R_2^{1/2} f(R_2^{-1/2} L_2 R_2^{-1/2}) R_2^{1/2} \\ &= cg(L_1,R_1) + (1-c)g(L_2,R_2). \end{aligned}$$

This inequality means that g is jointly convex. Conversely, let the perspective function g be jointly convex. It is clear that $f(L) = g(L,1)$. We have

$$\begin{aligned} f(cL_1 + (1-c)L_2) &= g(cL_1 + (1-c)L_2, 1) \\ &\le cg(L_1,1) + (1-c)g(L_2,1) \\ &= cf(L_1) + (1-c)f(L_2), \end{aligned}$$

where $L_1,L_2 \in H_n(I)$ and $0 \le c \le 1$. Hence, f is matrix convex.

Corollary 2.3. *The function f is matrix concave if and only if the perspective function g is jointly concave.*

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In ref. 1 for given continuous functions f and h, and commuting positive matrices L and R Edward G. Effros defined

$$(f\triangle h)(L,R) = f\left(\frac{L}{h(R)}\right)h(R),$$

and then proved the following theorem.

Theorem 2.4. *Suppose that f is matrix convex, $f(0) \le 0$ and that h is matrix concave with $h > 0$. Then $(L,R)\longmapsto(f\triangle h)(L,R)$ is jointly convex on positive commuting matrices L,R in the sense of Theorem 2.1.*

In the following, we prove that the jointly convexity of $f\triangle h$ implies the matrix convexity of f and the matrix concavity of h.

Theorem 2.5. *Suppose that f and h are continuous functions with $f(0) < 0$ and $h > 0$. Then f is matrix convex and h is matrix concave if and only if the generalized perspective function $f\triangle h$ is jointly convex.*

Proof: Let f be matrix convex and h be matrix concave. Let $L_1,L_2 \in H_n(\mathbb{I})$, $R_1,R_2 \in G_n(\mathbb{I})$, and $0 \le c \le 1$. Put $L =: cL_1 + (1-c)L_2$ and $R =: cR_1 + (1-c)R_2$. Define $T_1 =: (ch(R_1))^{1/2}h(R)^{-1/2}$ and $T_2 =: ((1-c)h(R_2))^{1/2}h(R)^{-1/2}$. The concavity of h implies that $T_1^*T_1 + T_2^*T_2 \le 1$. From matrix convexity of f and Theorem 2.1 of ref. 3, we have

$$\begin{aligned}(f\triangle h)(L,R) &= h(R)^{1/2}f(h(R)^{-1/2}Lh(R)^{-1/2})h(R)^{1/2}\\ &= h(R)^{1/2}f(T_1^*h(R_1)^{-1/2}L_1h(R_1)^{-1/2}T_1\\ &\quad + T_2^*h(R_2)^{-1/2}L_2h(R_2)^{-1/2}T_2)h(R)^{1/2}\\ &\le h(R)^{1/2}(T_1^*f(h(R_1)^{-1/2}L_1h(R_1)^{-1/2})T_1\\ &\quad + T_2^*f(h(R_2)^{-1/2}L_2h(R_2)^{-1/2})T_2)h(R)^{1/2}\\ &= ch(R_1)^{1/2}f(h(R_1)^{-1/2}L_1h(R_1)^{-1/2})h(R_1)^{1/2}\\ &\quad + (1-c)h(R_2)^{1/2}f(h(R_2)^{-1/2}L_2h(R_2)^{-1/2})h(R_2)^{1/2}\\ &= c(f\triangle h)(L_1,R_1) + (1-c)(f\triangle h)(L_2,R_2).\end{aligned}$$

Conversely, let the generalized perspective function $f\triangle h$ be jointly convex. It is clear that $f(L) = \frac{1}{h(1)}(f\triangle h)(h(1)L,1)$ and $h(R) = \frac{1}{f(0)}(f\triangle h)(0,R)$. Let $L_1,L_2 \in H_n(\mathbb{I})$ and $0 \le c \le 1$. Then,

$$\begin{aligned}f(cL_1 + (1-c)L_2) &= \frac{1}{h(1)}(f\triangle h)(h(1)(cL_1 + (1-c)L_2),1)\\ &= \frac{1}{h(1)}(f\triangle h)(c(h(1)L_1)\\ &\quad + (1-c)(h(1)L_2),1)\\ &\le \frac{1}{h(1)}(c(f\triangle h)(h(1)L_1,1)\\ &\quad + (1-c)(f\triangle h)(h(1)L_2,1))\\ &= \frac{1}{h(1)}(ch(1)f(L_1) + (1-c)h(1)f(L_2))\\ &= cf(L_1) + (1-c)f(L_2).\end{aligned}$$

This inequality means that f is matrix convex. Let $R_1,R_2 \in G_n(\mathbb{I})$ and $0 \le c \le 1$. Then,

$$\begin{aligned}(f\triangle h)(0,cR_1 + (1-c)R_2) \le c(f\triangle h)(0,R_1)\\ + (1-c)(f\triangle h)(0,R_2).\end{aligned}$$

Because $f(0) < 0$,

$$\begin{aligned}ch(R_1) + (1-c)h(R_2) &= \frac{1}{f(0)}(c(f\triangle h)(0,R_1)\\ &\quad + (1-c)(f\triangle h)(0,R_2))\\ &\le \frac{1}{f(0)}(f\triangle h)(0,cR_1 + (1-c)R_2)\\ &= h(cR_1 + (1-c)R_2).\end{aligned}$$

Hence, h is matrix concave.

Corollary 2.6. *Under the same hypotheses of Theorem 2.5, we have the following assertions:*

i. if f,h are matrix concave, then the generalized perspective function $f\triangle h$ is jointly concave.
ii. if the generalized perspective function $f\triangle h$ is jointly concave, then f is matrix concave and h is matrix convex.

1. Effros EG (2009) A matrix convexity approach to some celebrated quantum inequalities. *Proc Natl Acad Sci USA* 106:1006–1008.
2. Hansen F, Pedersen G (2003) Jensen's operator inequality. *B London Math Soc* 35:553–564.
3. Hansen F, Pedersen G (1982) Jensen's inequality for operators and Löwner's Theorem. *Math Ann* 258:229–241.