On the Waring problem for polynomial rings

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In this note we discuss an analog of the classical Waring problem for $\mathbb{C}[x_0,x_1,\ldots,x_n]$. Namely, we show that a general homogeneous polynomial $p\in\mathbb{C}[x_0,x_1,\ldots,x_n]$ of degree divisible by $k\geq 2$ can be represented as a sum of at most k^n k-th powers of homogeneous polynomials in $\mathbb{C}[x_0,x_1,\ldots,x_n]$. Noticeably, k^n coincides with the number obtained by naive dimension count.

sum of powers | sum of squares | Veronese embedding

We shall study a version of the general Waring problem for rings as posed in, e.g., ref. 1. Namely, we shall be concerned with the following problem.

Problem 1. *For any ring A and any integer $k>1$, let $A_k\subset A$ be the set of all sums of k-th powers in A. For any $a\in A_k$, let $w_k(a,A)$ be the least s such that a is a sum of s k-th powers. Determine $w_k(A)=\sup_{a\in A_k} w_k(a)$. (It is possible that $w_k(A)=\infty$).*

In many rings it makes sense to talk about generic elements in A_k and, similarly, one can ask to determine the number $\tilde{w}_k(A)=\sup_{a\in\tilde{A}_k} w_k(a)$, where $\tilde{A}_k$ is the appropriate set of generic elements in A_k. We will refer to the latter question as the "weak Waring problem" as opposed to Problem 1, which we call the "strong Waring problem."

Below we concentrate on $A=\mathbb{C}[x_0,x_1,\ldots,x_n]$ and for convenience work with homogeneous polynomials usually referred to as forms. In this case it is known that A_k coincides with the space of all forms in $\mathbb{C}[x_0,x_1,\ldots,x_n]$ whose degree is divisible by k. Thus, the strong Waring problem for $\mathbb{C}[x_0,x_1,\ldots,x_n]$ is formulated as follows. Denote by S_n^d the linear space of all forms of degree d in $n+1$ variables (with the 0-form included).

Problem 2. *Find the supremum over the set of all forms $f\in S_n^{kd}$ of the minimal number of forms of degree d needed to represent f as a sum of their k-th powers. In particular, how many forms of degree d are required to represent an arbitrary form $f\in S_n^{2d}$ as a sum of their squares?*

Recall that $\dim S_n^d=\binom{d+n}{n}$ and simple calculations show that

$$\frac{\dim S_n^{kd}}{\dim S_n^d}<k^n \quad\text{and}\quad \lim_{d\to\infty}\frac{\dim S_n^{kd}}{\dim S_n^d}=k^n.$$

Therefore, k^n is the lower bound for the answer to Problem 2. A version of Problem 2 related to the weak Waring problem is as follows.

Problem 3. *Find the minimum over all Zariski open subsets in S_n^{kd} of the number of forms of degree d needed to represent forms from these subsets as a sum of their k-th powers. In other words, how many k-th powers of forms of degree d are required to present a general form of degree kd?*

For sums of powers of linear forms a question very similar to Problem 3 was studied in greater detail by J. Alexander and A. Hirschowitz in the mid-1990's and was completely solved in a series of papers culminated in ref. 2; see also refs. 3 and 4. (This problem has a long history starting from the 19th century; see refs. 4 and 5, and it was later posed anew by H. Davenport.) In our notation the latter problem means that one fixes $d=1$ instead of letting d be an arbitrary positive integer and uses k as a parameter. The above-mentioned authors proved that the weak Waring problem for powers of linear forms has the solution expected by naive dimension count in all cases except for the case of quadrics in all dimensions, cubics in five variables and quartics in three, four, and five variables. On the other hand, their results and further investigations indicate that for $n>1$ the number of powers of linear terms required to present an arbitrary form of a given degree almost always exceeds the expected one obtained by naive dimension count; see, e.g., ref. 6, 1.6.

Our main result is the following.

Theorem 4. *Given a positive integer $k\geq 2$, then any general form f of degree kd in $n+1$ variables is a sum of at most k^n k-th powers. Moreover, for a fixed n this bound is sharp for all sufficiently large d.*

Thus k^n gives an upper bound for the answer to Problem 3 for any $n\geq 1$ and $k\geq 2$, and it is optimal for all sufficiently large d; see Remark 1 in *Final Remarks*.

Geometric Reformulation and Proof

For simplicity we work over $\mathbb{C}$, although our results hold for any algebraically closed field of characteristic zero. We refer to ref. 7 as a basic source of information on the geometry of tensors and its applications. The following result is classical, see, e.g., ref. 8.

Theorem 5. *(i) Any form f of even degree 2d in two variables is a sum of at most two squares;*

(ii) a general form of even degree 2d in two variables can be represented as a sum of two squares in exactly $\binom{2d-1}{d}$ ways.

The proof follows from the identity

$$f=A\cdot B=\left[\frac{1}{2}(A+B)\right]^2+\left[\frac{i}{2}(A-B)\right]^2$$

and $\binom{2d-1}{d}=\frac{1}{2}\binom{2d}{d}$ is the number of ways f can be presented as the product of two factors A and B of equal degree. Thus, for $n=1$ and $k=2$ the answer to Problem 2 is 2.

We recall that for any projective variety X, its p-th secant variety is defined as the Zariski closure of the union of the projective spans $\langle x_1,\ldots,x_p\rangle$ where $x_i\in X$. The following result gives a convenient reformulation of our problem.

Theorem 6. *Given a linear space V, a general polynomial in $S^{kd}V$ is a sum of p k-th powers $g_1^k,\ldots g_p^k$ where $g_i\in S^dV$ if and only if for p general forms $g_i\in S^dV$, $i=1,\ldots p$, the ideal generated by $g_1^{k-1},\ldots g_p^{k-1}$ contains $S^{kd}V$. (We shall call such an ideal kd-regular.)*

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Proof: The statement is a direct consequence of Terracini's lemma. Consider the subvariety X in the ambient space $\mathbb{P}S^{kd}V$ consisting of the k-th powers of all forms from S^dV. The tangent space to X at $g_i^k \in X$ is of the form $\{g_i^{k-1}f | f \in S^dV\}$. Therefore, the p-secant variety of X coincides with the ambient space $\mathbb{P}S^{kd}V$ if and only if the span of the tangent spaces to X at general g_i^k, (which is equal to $\{\sum_{i=1}^p g_i^{k-1}f_i | f_i \in S^dV\}$), coincides with $\mathbb{P}S^{kd}V$ as well. □

Theorem 6 relates Problem 3 to a special case of a conjecture of the first author about the Hilbert series of ideals generated by general forms in given degrees; see ref. 9.

We will show that if V is an $(n+1)$-dimensional linear space then the ideal generated by k^n general forms of the form g_i^{k-1} where $g_i \in S_n^dV$ is kd-regular, i.e., contains $S^{kd}V$.

To prove the latter claim it suffices to find k^n specific polynomials $\{g_1, \ldots, g_{k^n}\}$ of degree d such that the ideal generated by the powers g_i^{k-1} is kd-regular. Below, we will choose as g_i's powers of certain linear forms. For powers of linear forms one can use a new point of view related to apolarity. The space $T_{g^k}X^\perp$ orthogonal to $T_{g^k}X = \{g^{k-1}f | f \in S^dV\}$ is given by $T_{g^k}X^\perp = \{h \in S^{kd}V^\vee | h \cdot g^{k-1} = 0 \in S^dV^\vee\}$, i.e., is the space of polynomials in $V^\vee$ apolar to g^{k-1}. Moreover, when $g = l^m$, $l \in V$ the classical theory of apolarity provides a better result (for a recent reference see Lemma in ref. 5, p. 1094).

Proposition 7. *A form $f \in S^mV^\vee$ is apolar to l^{m-k}, i.e., $l^{m-k}f = 0$ if and only if all the derivatives of f of order $\le k$ vanish at $l \in V$.*

Using Proposition 7 one can reduce Theorem 4 to the following statement.

Theorem 8. *For a given integer $k \ge 2$ a form of degree kd in $(n+1)$ variables that has all derivatives of order $\le d$ vanishing at k^n general points vanishes identically.*

Our final effort will be to settle Theorem 8. Denote by $x_0, \ldots x_n$ a basis of V. Let $\xi_i = e^{2\pi i\sqrt{-1}/k}$ for $i = 0, \ldots, k-1$ be the (set of all) k-th roots of unity. By semicontinuity, it is enough to find k^n special points in $\mathbb{P}V \simeq \mathbf{P}^n$ such that a polynomial of degree kd in $\mathbf{P}^n$ that has all derivatives of order $\le d$ vanishing at these points must necessarily vanish identically. As such points we choose the points $(1, \xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_n})$ where $0 \le i_j \le k-1$, $1 \le j \le n$.

The following result proves even more than was claimed in Theorem 8.

Theorem 9. *For a given integer $k \ge 2$ a form of degree $kd + k - 1$ in $(n+1)$ variables that has all derivatives of order $\le d$ vanishing at k^n general points vanishes identically.*

Proof: As above we choose as our configuration the k^n points $(1, \xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_n})$ where $0 \le i_j \le k-1$, $1 \le j \le n$. Consider first the case $n = 1$. If a form $f(x_0, x_1)$ of degree $kd + k - 1$ has its derivatives of order $\le d$ vanishing at all $(1, \xi_i)$, then f should be divisible by $(x_1 - \xi_i x_0)^{d+1}$ for $i = 0, \ldots, k-1$. Therefore, if f is not vanishing identically, then its degree should be at least $k(d+1)$, which is a contradiction.

For $n \ge 2$ consider the arrangement of $\binom{n}{2}k$ hyperplanes given by $x_i = \xi_s x_j$ where $1 \le i < j \le n$, $0 \le s \le k-1$. One can easily check that this arrangement has the property that each hyperplane contains exactly k^{n-1} points and, furthermore, each point is contained in exactly $\binom{n}{2}$ hyperplanes. Indeed, consider, for example, the hyperplane $\mathcal{H}$ given by $x_n = \xi_i x_{n-1}$. The natural parametrization of $\mathcal{H}$ is by $(x_0, \ldots, x_{n-1}) \mapsto (x_0, x_1, \ldots, x_{n-1}, \xi_i x_{n-1})$ and the k^{n-1} points that lie on $\mathcal{H}$ correspond, according to this parametrization, exactly to $(1, \xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_{n-1}})$ for $0 \le i_j \le k-1$, $1 \le j \le n-1$. In other words, they correspond exactly to our arrangement of points in the previous dimension n. Our proof now proceeds by a double induction on the number of variables n and degree d. Assume that the statement holds for all d and up to n variables. (The case $n = 1$ is settled above.) Let us perform a step of induction in d. First we settle the case $d \le \binom{n}{2} - 1$. Consider a polynomial f of degree $kd + k - 1$ satisfying our assumptions. Restricting f to each of the above $\binom{n}{2}k$ hyperplanes $x_i = \xi_s x_j$ where $1 \le i < j \le n$, $0 \le s \le k-1$ we obtain the same situation in dimension n. By the induction hypothesis f vanishes on each such hyperplane and, therefore, must be divisible by H, where H is the product of the linear forms $x_i = \xi_s x_j$ defining all the chosen hyperplanes. (Obviously, $\deg H = \binom{n}{2}k$.) Thus, f vanishes identically because $k(\binom{n}{2} - 1) + k - 1 < \binom{n}{2}k$. For higher degrees we argue as follows. Take a form f of degree $kd + k - 1$ satisfying our assumptions. Restricting as above f to each of the above $\binom{n}{2}k$ hyperplanes $x_i = \xi_s x_j$ we obtain the same situation in dimension n. Again, by the induction hypothesis f vanishes on each such hyperplane and must be divisible by H. We get

$$f = H\tilde{f}$$

where $\deg \tilde{f} = k(d - \binom{n}{2}) + k - 1$ and $\tilde{f}$ has all derivatives of order $\le d - \binom{n}{2}$ vanishing at the same k^n points $(1, \xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_n})$. Indeed, in any affine coordinate system centered at any of these points, f has no terms of degree $\le d$. Because H has its lowest term in degree $\binom{n}{2}$ it follows that $\tilde{f}$ has no terms in degree $\le d - \binom{n}{2}$. By the induction hypothesis $\tilde{f}$ is identically zero. □

Notice that we have also obtained the following result of independent interest.

Corollary 10. *Any form of degree kd in $(n+1)$ variables can be expressed as a linear combination of the polynomials $(x_0 + \xi_{i_1}x_1 + \xi_{i_2}x_2 + \ldots + \xi_{i_n}x_n)^{(k-1)d}$ with coefficients being polynomials of degree d.*

Final Remarks

Remark 1: Although k^n is the correct asymptotic bound, it seems to be sharp only for considerably large values d. In particular, computer experiments show that for $k = 2$, $n = 3$, and $d \le 20$ seven general polynomials of degree d suffice to generate the space of polynomials in degree $2d$. All eight polynomials are required only for $d \ge 21$. Similarly, for $n = 4$ and $d \le 75$ experiments suggest that 15 (instead of the expected 16) general polynomials of degree d suffice to generate the space of polynomials in degree $2d$. Analogously, all 16 polynomials are required for $d \ge 76$. The ultimate challenge of this project is to solve completely Problem 3 for triples (n, k, d) and, in particular, to find the complete list of exceptional triples for which the answer to Problem 3 is larger than the one obtained by dimension count. Obviously, this list should include the list of exceptional cases obtained earlier by J. Alexander and A. Hirschowitz.

Remark 2: Theorem 4 seems to be new even in the classical case $k = 2$, i.e., for a sum of squares. In this case we have shown that any form of degree $2d$ in $(n+1)$ variables can be expressed as a linear combination of the polynomials $(x_0 \pm x_1 \pm x_2 + \ldots + \pm x_n)^d$ with coefficients being polynomials of degree d. Note that the former polynomials have real coefficients. But, obviously, our main result does not hold over the reals. It only implies that there is, in the usual topology, an open set of real polynomials of degree $2d$ that can be expressed as real linear combinations of 2^n squares of real polynomials of degree d. In other words, 2^n is a typical rank; see e.g., ref. 10. Notice that other typical ranks might also

MATHEMATICS

appear on other open subsets of polynomials. An example with three distinct typical ranks occurring in a similar situation can be found in ref. 10.

Remark 3: Although we only used powers of linear forms as the generators of the ideal in the above arguments it is not true that a general polynomial of degree kd can be expressed as a sum of at most k^n powers l_i^{kd} of linear forms l_i. For large d the number of necessary summands of the latter problem (solved by J. Alexander and A. Hirschowitz) equals $\lceil \frac{\binom{kd+n}{n}}{n+1} n+1 \rceil$. It grows as $(kd)^n/(n+1)!$ and is considerably larger than k^n.

Remark 4: Notice that the family of ideals generated by the powers of linear forms $(\xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2 + \ldots + \xi_n x_n)^d$ is a special case of ideals associated with hyperplane arrangements that appeared in several publications of the last decade; see e.g., refs. 11 and 12. In particular, it should be possible to calculate the Hilbert series of the quotient of the polynomial ring modulo these ideals, and the answer should be a certain specialization of the Tutte polynomial of the vector configuration given by the above linear forms, cf. ref. 12, section 5.

Remark 5: The above mentioned conjecture of the first author prescribes the Hilbert series of a homogeneous ideal generated by general forms of given degrees. Computer experiments show that the ideals generated by the powers of linear forms $(\xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2 + \ldots + \xi_n x_n)^d$ have, in general, another Hilbert series. On the other hand, it seems that in case $k = 2$ a different family of ideals generated by the powers of linear forms have the predicted Hilbert series. Namely, for every nonempty subset $I \subset \{0, \ldots n\}$ define $x_I = \sum_{i \in I} x_i$ and take $(x_I)^d$ for all subsets I with $|I|$ odd as generators of the ideal in question.

Remark 6: It is classically known that plane quartics can be represented as sums of three squares. It was recently observed in ref. 13 that the closure of the set of plane sextics that are sums of three squares forms a hypersurface of degree 83,200 in the space of all sextics.

Remark 7: Our results can be interpreted in the setting of osculating varieties. In notations of ref. 14 we have shown that any k-osculating space at the k^n-th secant variety of the kd-Veronese embedding of $\mathbb{P}^n$ fills out the ambient space.

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1. Gallardo L, Vaserstein L (2008) The strict Waring problem for polynomial rings. *J Number Theory* 128:2963–2972.
2. Alexander J, Hirschowitz A (1995) Polynomial interpolation in several variables. *J Alg Geom* 4:201–222.
3. Ciliberto C (2001) Geometric Aspects of Polynomial Interpolation in More Variables and of Waring's Problem. *Progr Math 201*, , European Congress of Mathematics (Birkhäuser, Basel), Barcelona 2000, Vol 1, pp 289–316.
4. Brambilla MC, Ottaviani G (2008) On the Alexander–Hirschowitz theorem. *J Pure Appl Alg* 212:1229–1251.
5. Iarrobino A (1995) Inverse systems of a symbolic power II. The Waring problem for forms. *J Algebra* 174:1091–1110.
6. Ranestad K, Schreyer FO (2000) Varieties of sums of powers. *J Reine Angew Math* 525:147–181.
7. Landsberg JM (2012) Tensors: Geometry and Applications. *Graduate Studies in Mathematics*, (AMS, Providence, RI), 128.
8. Choi M, Lam T, Reznick B (1995) Sums of squares of real polynomials. *Proc of Symp in Pure Math* 58(2):103–126.
9. Fröberg R (1985) An inequality for Hilbert series of graded algebras. *Math Scand* 56:117–144.
10. Comon P, Ottaviani G On the typical rank of real binary form. *Linear Multilinear Alg*, doi: 10.1080/03081087.2011.624097.
11. Postnikov A, Shapiro B (2004) Trees, parking functions, syzygies, and deformations of monomial ideals. *Trans Amer Math Soc* 356:3109–3142.
12. Ardila F, Postnikov A (2010) Combinatorics and geometry of power ideals. *Trans Amer Math Soc* 362:4357–4384.
13. Blekherman G, Hauenstein J, Ottem JC, Ranestad K, Sturmfels B Algebraic boundaries of Hilbert's SOS cones. arXiv:1107.1846.
14. Bernardi A, Catalisano MV, Gimigliano A, Idá M (2009) Secant varieties to osculating varieties of Veronese embeddings of $\mathbb{P}^n$. *J Algebra* 321:982–1004.