Surface phonons, elastic response, and conformal invariance in twisted kagome lattices

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Model lattices consisting of balls connected by central-force springs provide much of our understanding of mechanical response and phonon structure of real materials. Their stability depends critically on their coordination number $z$. $d$-dimensional lattices with $z = 2d$ are at the threshold of mechanical stability and are isostatic. Lattices with $z < 2d$ exhibit zero-frequency “floppy” modes that provide avenues for lattice collapse. The physics of systems as diverse as architectural structures, network glasses, randomly packed spheres, and biopolymer networks is strongly influenced by a near-isostatic lattice. We explore elasticity and phonons of a special class of two-dimensional isostatic lattices constructed by distorting the undistorted kagome lattice. We show that the phonon structure of these lattices, characterized by vanishing bulk moduli and thus negative Poisson ratios (equivalently, auxetic elasticity), depends sensitively on boundary conditions and on the nature of the kagome distortions. We construct lattices that under free boundary conditions exhibit surface floppy modes only or a combination of both surface and bulk floppy modes; and we show that bulk floppy modes present under free boundary conditions are also present under periodic boundary conditions but that surface modes are not. In the long-wavelength limit, the elastic theory of all these lattices is a conformally invariant field theory with holographic properties (characteristics of the bulk are encoded on the sample boundary), and the surface waves are Rayleigh waves. We discuss our results in relation to recent work on jammed systems. Our results highlight the importance of network architecture in determining floppy-mode structure.

Though the isostatic point always separates rigid from floppy behavior, the properties of isostatic lattices are not universal; rather they depend on lattice architecture. Here we explore the unusual properties of a particular class of periodic isostatic lattices derived from the two-dimensional kagome lattice by rigidly rotating triangles through an angle $\alpha$ without changing bond lengths as shown in Fig. 1. The bulk modulus $B$ of these lattices is rigorously zero for all $\alpha \neq 0$. As a result, their Poisson ratio acquires its limit value of $-1$; when stretched in one direction, they expand by an equal amount in the orthogonal direction: They are maximally auxetic (25–28). These modes represent collapse pathways (29, 30) of the kagome lattice. Modes of isostatic systems are generally very sensitive to boundary conditions (9, 31, 32), but the degree of sensitivity depends on the details of lattice structure. For reasons we will discuss more fully below, modes of the square lattice, which is isostatic, are in fact insensitive to changes from free boundary conditions to periodic boundary conditions (PBCs), whereas those of the undistorted kagome lattice are only mildly so. The modes of both, however, change significantly when rigid boundary conditions (RBCs) are applied. We show here that, in all families of the twisted kagome lattice, modes depend sensitively on whether FBCs, PBCs, or RBCs are applied. Finite lattices with free boundaries have floppy surface modes that are not present in their periodic or rigid spectrum or in that of finite undistorted kagome lattices. In the long-wavelength limit, the surface floppy modes, which are present in any $2d$ material with $B = 0$, reduce to surface Rayleigh waves (33) described by a conformally invariant energy whose analytic eigenfunctions are fully determined by boundary conditions. At shorter wavelengths, the surface waves become sensitive to lattice structure and remain confined to within a distance of the surface that diverges as the undistorted kagome lattice is approached. In the simplest twisted kagome lattice, all floppy modes are surface modes, but in more complicated lattices, including ones with uniaxial symmetry that we construct, there are both surface and bulk floppy modes.

Arguments due to Maxwell (1) provide a criterion for network stability: Networks in $d$ dimensions consisting of $N$ nodes, each connected with central-force springs to an average of $z$ neighbors, have $N_0 = dN - \frac{1}{2}zN$ zero-energy modes when $z < 2d$ (in the absence of redundant bonds—see below). Of these, a number, $N_{tr}$, which depends on boundary conditions, are trivial rigid translations and rotations, and the remainder are floppy modes of internal structural rearrangement. Under FBCs and PBCs, $N'_{tr}$ equals $d(d + 1)/2$ and $d$, respectively. With increasing $z$, mechanical stability is reached at the isostatic point at which $N_0 = N'_{tr}$.

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Fig. 1. (A) Section of a kagome lattice with \( N_x = N_y = 4 \) and \( N_z = N_xN_y \) three-site unit cells. Nearest-neighbor bonds, occupied by harmonic springs, are of length \( a \). The rotated row (second row from the top) represents a floppy mode. Next-nearest-neighbor bonds are shown as dotted lines in the lower left hexagon. The vectors \( e_1, e_2, \) and \( e_3 \) indicate symmetry directions of the lattice. The numbers in the triangles indicate those that twist together under PBCs in zero modes along the three symmetry direction. Note that there are only four of these modes. (B) Section of a square lattice depicting a floppy mode in which all sites along a line are displaced uniformly. (C) Twisted kagome lattice, with lattice constant \( a_l = 2a \cos \alpha \), derived from the undistorted lattice by rigidly rotating triangles through an angle \( \alpha \). A unit cell, bounded by dashed lines, is shown in yellow. Arrows depict site displacements for the zone-center (\( \{ 00 \} \) wavevector number \( \phi = 0 \) mode which has zero (nonzero) frequency under free (periodic) boundary conditions. Sites 1, 2, and 3 undergo no collective rotation about their center of mass, whereas sites 1, 2, and 3 do. (D) Superposed snapshots of the twisted lattice showing decreasing areas with increasing \( \alpha \).

The Maxwell argument is a global one; it does not provide information about the nature of the floppy modes and does not distinguish between bulk or surface modes.

Kagome Zero Modes and Elasticity

The kagome lattice of central-force springs shown in Fig. 1A is one of many locally isostatic lattices, including the familiar square lattice in two dimensions (Fig. 1B) and the cubic and pyrochlore lattices in three dimensions, with exactly \( z = 2d \) nearest-neighbor (NN) bonds connected to each site not at a boundary. Under PBCs, there are no boundaries, and every site has exactly 2d neighbors. Finite, \( N \)-site sections of these lattices have surface sites with fewer than \( 2d \) neighbors and of order \( \sqrt{N} \) zero modes. The free kagome lattice with \( N_x \) and \( N_y \) unit cells along its sides (Fig. 1A) has \( N = 3N_xN_y \) sites, \( N_B = 6N_xN_y - 2(N_x + N_y) + 1 \) bonds, and \( N_0 = 2(N_x + N_y) - 1 \) zero modes, all but three of which are floppy modes. These modes, depicted in Fig. 1A, consist of coordinated counterrotations of pairs of triangles along the symmetry axes \( e_1, e_2, \) and \( e_3 \) of the lattice. There are \( N_z \) modes associated with lines parallel to \( e_1, N_y \) associated with lines parallel to \( e_2, \) and \( N_y + N_z - 1 \) modes associated with lines parallel to \( e_3. \)

In spite of the large number of floppy modes in the kagome lattice, its longitudinal and shear Lamé coefficients, \( \lambda \) and \( \mu \), and its bulk modulus \( B = \lambda + \mu \) are nonzero and proportional to the NN spring constant \( k \): \( \lambda = \mu = \sqrt{3k}/8 \) and \( B = \lambda + \mu = \sqrt{3k}/4 \). The zero modes of this lattice can be used to generate an infinite number of distorted lattices with unstretched springs and thus zero energy (30). We consider only periodic lattices, the simplest of which are the twisted kagome lattices obtained by rotating triangles of the kagome unit cell through an angle \( \alpha \) as shown in Fig. 1C and D (30, 34). These lattices have \( C_6v \) rather than \( C_{3v} \) symmetry and, like the undistorted kagome lattice, three sites per unit cell. As Fig. 1D shows, the lattice constant of these lattices is \( a_l = 2a \cos \alpha \), and their area \( A_0 \) decreases as \( \cos^2 \alpha \) as \( \alpha \) increases. The maximum value that \( \alpha \) can achieve without bond crossings is \( \pi/3 \), so that the maximum relative area change is \( A_{2\alpha}/A_0 = 1/4 \). Because all springs maintain their rest length, there is no energy cost for changing \( \alpha \) and, as a result, \( B \) is zero for every \( \alpha \neq 0 \), whereas the shear modulus \( \mu = \sqrt{3k}/8 \) remains nonzero and unchanged. Thus, the Poisson ratio \( \sigma = (B - \mu)/(B + \mu) \) attains its smallest possible value of \(-1\). For any \( \alpha \neq 0 \), the addition of next-nearest-neighbor (NNN) springs, with spring constant \( k' \) (or of bending forces between springs) stabilizes zero-frequency modes and increases \( B \) and \( \sigma \). Nevertheless, for sufficiently small \( k' \), \( \sigma \) remains negative. Fig. 2 shows the region in the \( k'/k \) plane with negative \( \sigma \).

Kagome Phonon Spectrum

We now turn to the linearized phonon spectrum of the kagome and twisted kagome lattices subjected to PBCs. These conditions require displacements at opposite ends of the sample to be identical and thus prohibit distortions of the shape and size of the unit cell and rotations but not uniform translations, leaving two rather than three trivial zero modes. The spectrum (35) of the three lowest frequency modes along symmetry directions of the undistorted kagome lattice with and without NNN springs is shown in Fig. 3A. When \( k' = 0 \), there is a floppy mode for each wavenumber \( q \neq 0 \) running along the entire length of the three symmetry-equivalent straight lines running from \( M \) to \( \Gamma \) to \( M \) in the Brillouin zone (see Fig. 3, Inset). When \( N_x = N_y \), there are exactly \( N_x - 1 \) wavenumbers with \( q \neq 0 \) along each of these lines for a total of \( 3(N_x - 1) \) floppy modes. In addition, there are three zero modes at \( q = 0 \) corresponding to two rigid translations and one floppy mode that changes unit cell area at second but not first order in displacements, yielding a total of \( 3N_x \) zero modes rather than the \( 4N_x - 1 \) modes expected from the Maxwell count under FBCs. This discrepancy is our first indication of the importance of boundary conditions. The addition of NNN springs endows the floppy modes at \( k' = 0 \) with a characteristic frequency \( \omega_0 \sim \sqrt{k'} \) and causes them to hybridize with the acoustic phonon modes (Fig. 3A) (35). The result is an isotropic phonon spectrum up to wavenumber \( q^* = 1/2p^* \sim \sqrt{k'^2} \) and gaps at \( \Gamma \) and \( M \) of order \( \omega_0 \). Remarkably, at nonzero \( \alpha \) and \( k' = 0 \), the mode structure is almost identical to that at \( \alpha = 0 \) and \( k' > 0 \) with characteristic frequency \( \omega_0 \sim \sqrt{k^2} \sin \alpha \) and length \( l_\alpha \sim 1/\omega_0 \). In other words, twisting the kagome lattice through an angle \( \alpha \) has essentially the same effect on the spectrum as adding NNN springs with spring constant \( |\sin \alpha|^2 k \). Thus, under PBCs, the twisted kagome

![Fig. 2. Phase diagram in the \( \alpha - k' \) plane showing region with negative Poisson ratio \( \sigma \).](#)
lattice has no zero modes other than the trivial ones: It is "collectively" jammed in the language of refs. 32 and 36, but because it is not rigid with respect to changing the unit cell size, it is not strictly jammed.

Mode Counting and States of Self-Stress
To understand the origin of the differences in the zero-mode count for different boundary conditions, we turn to an elegant formulation (2) of the Maxwell rule that takes into account the existence of redundant bonds (i.e., bonds whose removal does not increase the number of floppy modes; ref. 13) and states in which springs can be under states of self-stress (3–6). Consider a ring network in two dimensions shown in Fig. 4 with \( N = 4 \) nodes and \( N_b = 4 \) springs with three springs of length \( a \) and one spring of length \( b \). The Maxwell count yields \( N_\theta = 4 = 3 + 1 \) zero modes: two rigid translations, one rigid rotation, and one internal floppy mode—all of which are "finite-amplitude" modes with zero energy even for finite-amplitude displacements. When \( b = 3a \), the Maxwell rule breaks down. In the zero-energy configuration, the long spring and the three short ones are collinear, and a prestressed state in which the \( b \) spring is under compression and the three \( a \) springs are under tension (or vice versa) but the total force on each node remains zero becomes possible. This is called a state of self-stress. The system still has three finite-amplitude zero modes corresponding to arbitrary rigid translations and rotations, but the finite-amplitude floppy mode has disappeared. In the absence of prestress, it is replaced by two "infinitesimal" floppy modes of displacements of the two internal nodes perpendicular of the now linear network. In the presence of prestress, these two modes have a frequency proportional to the square root of the tension in the springs. Thus, the system now has one state of self-stress and one extra zero mode in the absence of prestress, implying \( N_\theta = 2N - N_B + S \), where \( S \) is the number of states of self-stress.

This simple count is more generally valid, as can be shown with the aid of the equilibrium and compatibility matrices (2), denoted, respectively, as \( H \) and \( C \equiv H^T \). \( H \) relates the vector \( t \) of \( N_B \) spring tensions to the vector \( f \) of \( dN \) forces at nodes via \( H \cdot t = f \), and \( C \) relates the vector \( d \) of \( dN \) node displacements to the vector \( e \) of \( N_b \) spring stretches via \( C \cdot d = e \). The dynamical matrix determining the phonon spectrum is \( D = \lambda H \cdot H^T \).

Vectors \( t_0 \) in the null space of \( H \) (\( H \cdot t_0 = 0 \)), describe states of self-stress, whereas vectors \( d_0 \) in the null space of \( C \) represent displacements with no stretch e—i.e., modes of zero energy. Thus the null-space dimensions of \( H \) and \( C \) are, respectively, \( S \) and \( N_\theta \). The rank-nullity theorem of linear algebra (37) states that the rank \( r \) of a matrix plus the dimension of its null space equals its column number. Because the rank of a matrix and its transpose are equal, the \( H \) and \( C \) matrices, respectively, yield the relations \( r + S = N_B \) and \( r + N_\theta = dN \), implying \( N_\theta = dN - N_B + S \). Under PBCs, locally isostatic lattices have \( Z = 2d \) exactly, and the Maxwell rule yields \( N_\theta = 0 \): There should be no zero modes at all. But we have just seen that both the square and undistorted kagome lattices under PBCs have of \( \sqrt{N} \) zero modes as calculated from the dynamical matrix, which, because it is derived from a harmonic theory, does not distinguish between infinitesimal and finite-amplitude zero modes. Thus, in order for there to be zero modes, there must be states of self-stress, in fact, one state of self-stress for each zero mode.

In the square lattice under FBCs, \( N = N_\theta \) and \( N_B = 2N_\theta \). There are \( 2N_\theta \) zero states of self-stress and \( N_\theta \) zero modes depicted in Fig. 1B. Under PBCs, the dimension of the null space of \( H \) is \( S = N_\theta + N_\theta \), and there are also \( N_\theta = S = N_\theta \) zero modes that are identical to those under FBCs. We have already seen that there are \( N_\theta = 2(N_\theta + N_\theta) - 1 \) zero modes in the free undistorted kagome lattice. Direct evaluations (29) (see SI Text) of the dimension of the null spaces of \( H \) and \( C \) for the undistorted kagome lattice with PBCs yields \( S = N_\theta = 3N_\theta \). The zero modes under PBCs are identical to those under FBCs except that the \( 2N_\theta - 1 \) modes associated with lines parallel to \( e_2 \) under FBCs get reduced to \( N_\theta \) modes because of the identification of opposite sides of the lattice required by the PBCs, as shown in Fig. 1A. Thus the modes of both the square and kagome lattices do not depend strongly on whether FBCs or PBCs are applied. Under RBCs, however, the floppy modes of both disappear. The situation for the twisted kagome lattice is different. There are still \( 2(N_\theta + N_\theta) - 1 \) zero modes under FBCs, but there are only two states of self-stress under PBCs and thus only \( N_\theta = 2 \) zero modes, as a direct evaluation of the null spaces of \( H \) and \( C \) verifies (SI Text), in agreement with the results obtained via direct evaluation of the eigenvalues of the dynamical matrix (35, 38). All of the floppy modes under FBCs have disappeared.

Effective Theory and Edge Modes
An effective long-wavelength energy \( E_{\text{eff}} \) for the low-energy acoustic phonons and nearly floppy distortions provides insight into the nature of the modes of the twisted kagome lattice. The variables in this theory are the vector displacement field \( u(x) \) of nodes at undistorted positions \( x \) and the scalar field \( \phi(x) \) describing nearly floppy distortions within a unit cell. The detailed form of \( E_{\text{eff}} \) depends on which three lattice sites are assigned to a unit cell. Fig. 1C depicts the lattice distortion \( \phi \) for the nearly floppy mode at \( \Gamma \) (with energy proportional to \( \sin^2 \alpha \)) along with a particular representation of a unit cell, consisting of a central asymmetric hexagon and two equilateral triangles, with eight sites on its boundary. If sites 1, 2, and 3 are assigned to the unit cell, then the distortion \( \phi \) involves no rotations of these sites relative to their center of mass, and the harmonic limit of \( E_{\text{eff}} \) depends only on the symmetrized and linearized strain \( u_{ij} = (\partial_i u_j + \partial_j u_i)/2 \) and on \( \phi \):

\[
E = \frac{1}{2} \int d^2x \left[ \mu \ddot{\phi}^2 + K(\phi + \dddot{\phi})^2 + V(\partial_i \phi)^2 - WT_{ij} \dot{u}_j \partial_i \phi \right],
\]

where \( \dddot{\phi} = u_{ij} - \frac{1}{2} \dddot{\phi} \eta_{kk} \) is the symmetric-traceless stain tensor, \( \mu = \sqrt{3k}/8 \), \( K = 3\sqrt{3} \tan^2 \alpha/a^2 \), \( \dddot{\phi} = a \cos(\alpha/2\sqrt{3}) \), \( W = \sqrt{3k}/4 + O(a^2) \), and \( V = \sqrt{3k}/8 + O(a^2) \). The last term in

\[\text{\textit{Fig. 3.}}\ (A)\ Phonon\ spectrum\ for\ the\ undistorted\ kagome\ lattice.\ Dashed\ lines\ depict\ frequencies\ at\ } k' = 0 \text{ and full lines at } k' = 0.\ \text{The inset shows the Brillouin zone with symmetry points } \Gamma, M, \text{ and } K.\ \text{Note the line of zero modes along } \Gamma M \text{ when } k' = 0, \text{ all of which develop nonzero frequencies for wavevector } q > 0 \text{ when } k' > 0 \text{ reaching } q \approx \sqrt{k'} \text{ on a plateau beginning at } q = q_0 \approx 1/q'.\ \text{(B) Phonon spectrum for } \alpha > 0 \text{ and } k' = 0.\ \text{The plateau along } \Gamma M \text{ defines } \omega_0 \approx \sqrt{k' \sin \alpha} \text{ and its onset at } q_0 \approx \omega_0 \text{ defines a length } l_q \approx 1/\sin \alpha.\]

\[\text{\textit{Fig. 4.}}\ (A)\ Ring\ network\ with\ \( b > 3a \)\ showing\ internal\ floppy\ mode.\ \text{(B) Ring-network with } b = 3a \text{ showing one of the two infinitesimal modes.}\]
which $\Gamma^{ijk}$ is a third-rank tensor, whose only nonvanishing components are $\Gamma^{axx} = -\Gamma^{oxy} = -\Gamma^{xoy} = \Gamma^{oxy} = -\Gamma^{axy} = -\Gamma^{xoy} = -\Gamma^{axy}$, is invariant under operations of the group $C_n$ but not under arbitrary rotations. The $Kx\xi_k\theta_i(x)$ term is the only one that reflects the $C_n$ (rather than $C_{2n}$) symmetry of the lattice. There are several comments to make about this energy. The gauge-like coupling in which the isotropic strain $u_i$ appears only in the combination $(\phi + 2\mu_0 u_i)$ guarantees that the bulk modulus vanishes: $\phi$ will simply relax to $-2\mu_0 u_i$ to reduce to zero the energy of any state with nonvanishing $u_i$. The coefficient $K$ can be calculated directly from the observation that, under $\phi$ alone, the length of every spring changes by $\delta a = -3\beta a \sin \alpha$, and this length change is reversed by a homogeneous volume change $u_i = 2\delta a_0 / a_0 = -2\delta a / a$. In the $\alpha \to 0$ limit, $K \to 0$, and the energy reduces to that of an isotropic solid with bulk modulus $K_0 = \lim_{\alpha \to 0} K^{a_2} = \frac{3}{4}k/4$ if the $V$ and $W$ terms, which are higher order in gradients, are ignored. The $W$ term gives rise to a term, singular in gradients of $u$, when $\phi$ is integrated out that is responsible for the finite-wavenumber elastic energy from isotropy. At small $\alpha$, the length scale $l_\alpha$ appears in several places in this energy: in the length $\xi = l_\alpha$ and in the ratios $\sqrt{\mu / K}, \sqrt{\nu / K}$, and $\sqrt{\nu / W}$. At length scales much larger than $l_\alpha$, the $V$ and $W$ terms can be ignored, and $\phi$ relaxes to $-2\mu_0 u_i$, leaving only the shear elastic energy of an elastic solid proportional to $\mu_\alpha$. At length scales shorter than $l_\alpha$, $\phi$ deviates from $-2\mu_0 u_i$ and contributes significantly to the form of the energy spectrum. If $1, 2^\ast$, and $3^\ast$ in Fig. 1D are assigned to the unit cell, then $\phi$ involves rotations relative to the lattice axes, and the energy develops a Cosserat-like form (39, 40), that is a function of $\phi = a(V \times u_j) / 2$ rather than.

The modes of our elastic energy in the long-wavelength limit ($q_{ax} \ll 1$) are simply those of an elastic medium with $B = 0$. In this limit, there are transverse and longitudinal bulk sound modes with equal sound velocities $c_T = \sqrt{\mu / \rho} = (a / 2) / \sqrt{k / m}$ and $c_L = \sqrt{(B + \mu) / \rho}$, which is the particle mass at each node and $\rho$ is the mass density. In addition there are Rayleigh surface waves (33) in which there is a single decay length (rather than the two at $B > 0$), and displacements are proportional to $e^{-k_0 \cos \theta x}$ with $q_{y} = q_{z}$ for a semiinfinite sample in the upper half-plane so that the penetration depth into the interior is $1 / q_x$. These waves have zero frequency in two dimensions when $B = 0$, and they do not appear in the spectrum with PBCs. Thus this simple continuum limit provides us with an explanation for the difference between the spectrum of the free and periodic twisted kagome lattices. Under FBCs, there are zero-frequency surface modes not present under PBCs.

Further insight into how boundary conditions affect spectrum follows from the observation that the continuum elastic theory with $B = 0$ depends only on $u_i$. The metric tensor $g_{ij}(x)$ of the distorted lattice is related to the strain $u_i(x)$ via the simple relation $g_{ij}(x) = \delta_{ij} + 2u_{ij}(x)$; and $u_{ij}(x) = \frac{1}{2} (\delta_{ij} - g_{ij}(x) / 2$, which is zero for $g_{ij} = \delta_{ij}$, is invariant, and thus remains equal to zero, under conformal transformations that take the metric tensor from its reference form $\delta_{ij}$ to $h(x)\delta_{ij}$ for any continuous function $h(x)$. The zero modes of the theory thus correspond simply to conformal transformations, which in two dimensions are best represented by the complex position and displacement variables $z = x + iy$ and $w(z) = u(x) + u(y)$. All conformal transformations are described by an analytic displacement field $w(z)$. Because, by Cauchy’s theorem, analytic functions in the interior of a domain are determined entirely by their values on the domain’s boundary (the “holomorphic” property; ref. 41), the zero modes of a given sample are simply those analytic functions that satisfy its boundary conditions. For example, a disc with fixed edges ($u = 0$) has no zero modes because the only analytic function satisfying this FBC is the trivial one $w(z) = 0$; but a disc with free edges (stress and thus strain equal to zero) has one zero mode for each of the analytic functions $w(z) = a_{m}z^{n}$ for integer $n \geq 0$. The boundary conditions $\lim_{y \to \pm \infty} u(x, y) = 0$ and $u(x, y) = u(x + L, y)$ on a semiinfinite cylinder with axis along $x$ are satisfied by the function $w(z) = a_{0}^{y} = e^{i \pi \sigma}e^{-\alpha z}$ when $q_{z} = 2\pi n / L$, where $n$ is an integer. This solution is identical to that for classical Rayleigh waves on the same cylinder. Like the Rayleigh theory, the conformal theory puts no restriction on the value of $n$ (or equivalently $q_{z}$). Both theories break down, however, at $q_{z} = q_{1} \approx \min(l_{\alpha}^{-1}, a^{-1}-1$, beyond which the full lattice theory, which yields a complex value of $q_{z} = q_{s} + iq_{c}$, is needed.

Fig. 5A shows an example of a surface wave. At the bottom of this figure, $u_{x}(x)$ is an almost perfect sinusoid. As $y$ decreases toward the surface, the amplitude grows, and in this picture reaches the nonlinear regime by the time the surface at $y = 0$ is reached. Fig. 5B plots $q_{y}^{2}$ as a function of $\alpha$, obtained both by direct numerical evaluation and by an analytic transfer matrix procedure (42) for different values of $\alpha$ (SI Text). The Rayleigh limit $q_{y}^{2} = q_{s}^{2}$ is reached for all $\alpha$ as $q_{s} \to 0$. Interestingly, the Rayleigh limit remains a good approximation up to values of $q_{s}$ that increase with increasing $\alpha$. The inset to Fig. 5 plots $q_{s}L_{a}$ as a function of $\eta = q_{s}L_{a}$, and shows that in the limit $\alpha \to 0$ ($L_{a} / a \to \infty$), $q_{s}^{2}$ obeys an $\alpha$-independent scaling law of the form $q_{s}^{2} = q_{0}^{2}$. The full complex $q_{y}$ obeys a similar equation. This type of behavior is familiar in critical phenomena where scaling occurs when correlation lengths become much larger than microscopic lengths. The function $f(\eta)$ approaches $\eta$ as $\eta \to 0$ and asymptotes to $4 / 3$ for $\eta \to \infty$. Thus for $q_{s}L_{a} \ll 1, q_{s} = q_{0}$, and for $q_{s}L_{a} \gg 1, q_{s} = (4 / 3)q_{0}^{-1}$. As $\alpha$ increases, $L_{a} / a$ is no longer much larger than one, and deviations from the scaling law result. The situation for

![Fig. 5](image-url)
A lattice with point group symmetry. Its geometry has uniaxial symmetries related to continuous curves at large values of wavenumber. The spectrum is absolutely frequency branch of the uniaxial kagome lattice. The spectrum is absolutely frequency branch of the uniaxial kagome lattice. The spectrum is absolutely frequency branch of the uniaxial kagome lattice. The spectrum is absolutely frequency branch of the uniaxial kagome lattice. The spectrum is absolutely frequency branch of the uniaxial kagome lattice. The spectrum is absolutely frequency branch of the uniaxial kagome lattice. The spectrum is absolutely frequency branch of the uniaxial kagome lattice.

Connections to Other Systems

Our study highlights the rich and remarkable variety of physical properties that isostatic systems can exhibit. Under FBCs, floppy modes can adopt a variety of forms, from all being extended to all being localized near surfaces to a mixture of the two. Under PBCs, the presence of floppy modes depends on whether the lattice can or cannot support states of self-stress. When a lattice exhibits a large number of zero-energy edge modes, its mechanical/dynamical properties become extremely sensitive to boundary conditions, much as do the electronic properties of the topological states of matter studied in quantum systems (47–50). The zero-energy edge modes observed in our isostatic lattices are collective modes whose amplitudes decay exponentially from the edge with a finite decay length, in direct contrast to the very localized and trivial floppy modes arising from dangling bonds.

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