

Euler and Navier–Stokes equations on the hyperbolic plane

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We show that nonuniqueness of the Leray–Hopf solutions of the Navier–Stokes equation on the hyperbolic plane \mathbb{H}^2 observed by Chan and Czubak is a consequence of the Hodge decomposition. We show that this phenomenon does not occur on \mathbb{H}^n whenever $n \geq 3$. We also describe the corresponding general Hamiltonian framework of hydrodynamics on complete Riemannian manifolds, which includes the hyperbolic setting.

harmonic forms | steady flows | ill-posedness | Dirichlet problem | Dodziuk's theorem

Consider the initial value problem for the Navier–Stokes equations on a complete n -dimensional Riemannian manifold M

$$\partial_t v + \nabla_v v - Lv = -\text{grad } p, \quad \text{div } v = 0 \quad [1]$$

$$v(0, x) = v_0(x). \quad [2]$$

The symbol ∇ denotes the covariant derivative and $L = \Delta - 2r$, where Δ is the Laplacian on vector fields and r is the Ricci curvature of M . Dropping the linear term Lv from the first equation in Eq. 1 yields the Euler equations of hydrodynamics,

$$\partial_t v + \nabla_v v = -\text{grad } p, \quad \text{div } v = 0. \quad [3]$$

Most of the work on well-posedness of the Navier–Stokes equations has focused on the cases where M is either a domain in \mathbb{R}^n or the flat n -torus \mathbb{T}^n . In fundamental contributions J. Leray and E. Hopf established existence of an important class of weak solutions described as those divergence-free vector fields v in $L^\infty([0, \infty), L^2) \cap L^2([0, \infty), H^1)$ that solve the Navier–Stokes equations in the sense of distributions and satisfy

$$\|v(t)\|_{L^2}^2 + 4 \int_0^t \|\text{Def } v(s)\|_{L^2}^2 ds \leq \|v_0\|_{L^2}^2 \quad [4]$$

$$\lim_{t \rightarrow 0} \|v(t) - v_0\|_{L^2} = 0$$

for any $0 \leq t < \infty$ and where $\text{Def } v = \frac{1}{2}(\nabla v + \nabla v^T)$ is the so-called deformation tensor (ref. 2). When $n = 2$ using interpolation inequalities and energy estimates, it is possible to show that the Leray–Hopf solutions are unique and regular but the problem is in general open for $n = 3$ (e.g., refs. 3 and 4).

There have also been studies on curved spaces, which with few exceptions have been confined to compact manifolds (possibly with boundary) (e.g., ref. 5 and references therein). In a recent paper (1) Chan and Czubak studied the Navier–Stokes equation on the hyperbolic plane \mathbb{H}^2 and more general noncompact manifolds of negative curvature. In particular, they showed that in the former case the Cauchy problem (Eq. 1 and 2) admits nonunique Leray–Hopf solutions and in the latter a similar nonuniqueness

holds for a modified Navier–Stokes equation using the results of Anderson (6) and Sullivan (7).

Our goal in this paper is to provide a direct formulation of the nonuniqueness of the Leray–Hopf solutions on \mathbb{H}^2 and explain that it relies on the specific form of the Hodge decomposition for 1-forms (or vector fields) in this case. We also show that no such phenomenon can occur in the hyperbolic space \mathbb{H}^n with $n \geq 3$, thus answering the question raised in ref. 1. As a by-product, we describe the corresponding Hamiltonian setting of the Euler equations on complete Riemannian manifolds (in particular, hyperbolic spaces).

We point out that this type of nonuniqueness cannot be found in the Euler equations. Furthermore, it is of a different nature than the examples constructed, e.g., by Shnirelman (8) or De Lellis and Székelyhidi (9). On the other hand, it is similar to nonuniqueness of solutions of the Navier–Stokes equations defined in unbounded domains of the higher-dimensional Euclidean space (cf. Heywood, ref. 10).

1. Stationary Harmonic Solutions of the Euler Equations

Our main result is summarized in the following theorem.

Theorem 1.1.

- i) There exists an infinite-dimensional space of stationary L^2 harmonic solutions of the Euler equations on \mathbb{H}^2 .
- ii) There are no stationary L^2 harmonic solutions of the Euler equations on \mathbb{H}^n for any $n > 2$.

Proof: Recall the Hamiltonian formulation of the Euler Eq. 3 on a complete Riemannian manifold M (e.g., ref. 11). Consider the Lie algebra $\mathfrak{g}_{\text{reg}} = \text{Vect}_\mu(M)$ of (sufficiently smooth) divergence-free vector fields on M with finite L^2 norm. Its dual space $\mathfrak{g}_{\text{reg}}^*$ has a natural description as the quotient space $\Omega_{L^2}^1/d\Omega_{L^2}^0$ of the L^2 1-forms modulo (the L^2 closure of) the exact 1-forms on M . Namely, the pairing between cosets $[\beta] \in \Omega_{L^2}^1/d\Omega_{L^2}^0$ of 1-forms $\beta \in \Omega_{L^2}^1$ and vector fields $w \in \text{Vect}_\mu(M)$ is given by

$$\langle [\beta], w \rangle := \int_M (\iota_w \beta) d\mu,$$

where ι_w is the contraction of a differential form with a vector field w , and μ is the Riemannian volume form on M .

Let $A: \mathfrak{g}_{\text{reg}} \rightarrow \mathfrak{g}_{\text{reg}}^*$ denote the inertia operator defined by the Riemannian metric. The operator A assigns to a vector field $v \in \text{Vect}_\mu(M)$ the coset $[v^b]$ of the corresponding 1-form v^b via the pairing given by the metric. The coset is defined as the 1-form up

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to adding differentials of the L^2 functions on M . Thus, in the Hamiltonian framework the Euler equation reads

$$\frac{d}{dt}[v^b] = -L_v[v^b],$$

where $[v^b] \in \overline{\Omega_{L^2}^1/d\Omega_{L^2}^0}$ and L_v is the Lie derivative in the direction of the vector field v .

The space $\Omega_{L^2}^1$ of the L^2 1-forms on a complete manifold M admits the Hodge–Kodaira decomposition

$$\Omega_{L^2}^1 = \overline{d\Omega_{L^2}^0} \oplus \overline{\delta\Omega_{L^2}^2} \oplus \mathcal{H}_{L^2}^1,$$

where the first two summands denote the L^2 closures of the images of the operators d and δ , whereas $\mathcal{H}_{L^2}^1$ is the space of the L^2 harmonic 1-forms on M . Therefore, we have a natural representation of the dual space

$$\mathfrak{g}_{\text{reg}}^* = \overline{\delta\Omega_{L^2}^2} \oplus \mathcal{H}_{L^2}^1.$$

It turns out that the harmonic forms in the above representation correspond to steady solutions of the Euler equation. Namely, one has the following proposition.

Proposition 1.2. *Each harmonic 1-form on a complete manifold M that belongs to $L^2 \cap L^4$ defines a steady solution of the Euler Eq. 3 on M .*

Proof of Proposition 1.2: Let α be a bounded L^2 harmonic 1-form on M . Let v_α denote the divergence-free vector field corresponding to α ; i.e., $v_\alpha^b = \alpha$. Because the 1-form α is harmonic, using Cartan’s formula gives

$$L_{v_\alpha}\alpha = i_{v_\alpha}d\alpha + dt_{v_\alpha}\alpha = dt_{v_\alpha}\alpha.$$

We claim that $t_{v_\alpha}\alpha \in \Omega_{L^2}^0$ and consequently $dt_{v_\alpha}\alpha \in d\Omega_{L^2}^0$. Indeed, by the definition of the vector field v_α we have

$$\|t_{v_\alpha}\alpha\|_{L^2}^2 = \int_M (\alpha(v_\alpha))^2 d\mu = \|\alpha\|_{L^4}^4,$$

which is finite by assumption. It follows that the 1-form $dt_{v_\alpha}\alpha$ must correspond to the zero coset in the quotient space $\mathfrak{g}_{\text{reg}}^* = \Omega_{L^2}^1/d\Omega_{L^2}^0$, which in turn implies that $L_{v_\alpha}\alpha = 0 \in \mathfrak{g}_{\text{reg}}^*$. The latter means that the 1-form α defines a steady solution of the Euler equation, $d\alpha/dt = -L_{v_\alpha}\alpha = 0$, which proves the proposition.

If M is compact, then the space of harmonic 1-forms is always finite-dimensional [and isomorphic to the deRham cohomology group $H^1(M)$].

According to a result of Dodziuk (12), the hyperbolic space \mathbb{H}^n carries no L^2 harmonic k -forms except for $k = n/2$, in which case it is infinite-dimensional. It therefore follows that there can be no L^2 harmonic stationary solutions of the Euler equations on \mathbb{H}^n for any $n > 2$, which proves part *ii* of *Theorem 1.1*.

To prove part *i* we note that for $n = 2$ the space of harmonic 1-forms on \mathbb{H}^2 is infinite-dimensional. Moreover, it allows for the following construction. Consider the subspace $\mathcal{S} \subset \mathcal{H}_{L^2}^1$ of 1-forms that are differentials of bounded harmonic functions whose differentials are both in L^2 and L^4 :

$$\mathcal{S} = \{d\Phi \mid \Phi \text{ is harmonic on } \mathbb{H}^2 \text{ and } d\Phi \in L^2 \cap L^4\}.$$

It turns out that the subspace \mathcal{S} is already infinite-dimensional. Indeed, let us consider the Poincaré model of \mathbb{H}^2 , i.e., the unit disk \mathbb{D} with the hyperbolic metric $\langle \cdot, \cdot \rangle_h$, which we denote by \mathbb{D}_h . It is conformally equivalent to the standard unit disk with the

Euclidean metric $\langle \cdot, \cdot \rangle_e$, denoted by \mathbb{D}_e . Bounded harmonic functions on \mathbb{D}_h can be obtained by solving the Dirichlet problem on \mathbb{D}_e , i.e., by constructing harmonic functions Φ on \mathbb{D} with boundary values φ prescribed on $\partial\mathbb{D}$. First, note that any such 1-form $d\Phi$ is clearly harmonic:

$$\Delta d\Phi = d\delta d\Phi = d\Delta\Phi = 0.$$

Second, observe that

$$\begin{aligned} \|d\Phi\|_{L^2(\mathbb{D}_h)}^2 &= \int_{\mathbb{D}} \langle d\Phi, d\Phi \rangle_h d\mu_h = \int_{\mathbb{D}} \det(g^{ij}) \langle d\Phi, d\Phi \rangle_e \det(g_{ij}) d\mu_e \\ &= \int_{\mathbb{D}} \langle d\Phi, d\Phi \rangle_e d\mu_e = \|d\Phi\|_{L^2(\mathbb{D}_e)}^2 \end{aligned}$$

and

$$\begin{aligned} \|d\Phi\|_{L^4(\mathbb{D}_h)}^4 &= \int_{\mathbb{D}} \langle d\Phi, d\Phi \rangle_h^2 d\mu_h = \int_{\mathbb{D}} \det^2(g^{ij}) \langle d\Phi, d\Phi \rangle_e^2 \det(g_{ij}) d\mu_e \\ &= \int_{\mathbb{D}} (1 - |z|^2)^2 \langle d\Phi, d\Phi \rangle_e^2 d\mu_e(z) \leq \int_{\mathbb{D}} \langle d\Phi, d\Phi \rangle_e^2 d\mu_e \\ &= \|d\Phi\|_{L^4(\mathbb{D}_e)}^4, \end{aligned}$$

where $\det(g_{ij}) = 1/(1 - |z|^2)^2$ is the determinant of the hyperbolic metric.

Furthermore, for sufficiently smooth boundary values $\varphi \in C^{1+\sigma}(\partial\mathbb{D})$ there is a uniform upper bound for its harmonic extension inside the disk,

$$|d\Phi(x)| \leq C \|\varphi\|_{C^{1+\sigma}(\partial\mathbb{D})}$$

for any $x \in \mathbb{D}$ and $0 < \sigma < 1$, and some positive constant C (e.g., ref. 13). This implies that (for sufficiently smooth φ) the 1-forms $d\Phi$ define an infinite-dimensional subspace \mathcal{S} of harmonic forms in $L^2 \cap L^4$, which satisfy assumptions of the proposition above. It follows that they define an infinite-dimensional space of stationary solutions of the Euler equations on the hyperbolic plane \mathbb{H}^2 . This completes the proof of *Theorem 1.1*.

2. Nonunique Leray–Hopf Solutions of the Navier–Stokes Equations

Using the fact that suitably scaled steady solutions of the Euler equations also solve the Navier–Stokes system, the authors in ref. 1 obtained a type of ill-posedness result for the Leray–Hopf solutions in the hyperbolic setting.

Theorem 2.1 (ref. 1). *Given a vector field $v_e = (d\Phi)^\#$ on \mathbb{H}^2 there exist infinitely many real-valued functions $f(t)$ for which $v_{ns} = f(t)v_e$ is a weak solution of the Navier–Stokes equations with decreasing energy (i.e., satisfying the Leray–Hopf conditions).*

An immediate consequence of this result and *Theorem 1.1* is the following.

Corollary 2.2. *There exist infinitely many weak Leray–Hopf solutions to the Navier–Stokes equation on \mathbb{H}^2 . There are no nonunique Leray–Hopf harmonic solutions to the Navier–Stokes equation on \mathbb{H}^n with $n \geq 3$ arising from the above construction.*

Remark 2.3: The phenomenon of nonuniqueness of solutions to the Navier–Stokes equation in unbounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is of similar nature (10). Indeed, that construction is based on the existence of a harmonic function with gradient in L^2 and appropriate boundary conditions in such domains. The Green

function $\Phi(x) = G(a, x)$ centered at a point a outside of Ω has the decay like $G(a, x) \sim |x|^{2-n}$ as $x \rightarrow \infty$, so that $|d\Phi(x)| \sim |x|^{1-n}$ and hence $|d\Phi(x)|^2 \sim |x|^{2-2n}$. Thus, for $n \geq 3$ the 1-forms $d\Phi$ belong to $L^2 \cap L^4$ on Ω . The corresponding divergence-free vector fields $(d\Phi)^\sharp$ provide examples of stationary Eulerian solutions in Ω (with nontrivial boundary conditions) and can be used to construct time-dependent weak solutions $v_{ns} = f(t)(d\Phi)^\sharp$ to the Navier–Stokes equation in Ω , as in *Theorem 2.1*.

3. Appendix

To make this paper self-contained we provide here some details of the construction of the weak solutions given in ref. 1. It is convenient to rewrite the Navier–Stokes Eq. 1 in the language of differential forms,

$$\partial_t v^b + \nabla_v v^b - \Delta v^b + 2r(v^b) = -dp, \quad \delta v^b = 0, \quad [5]$$

where $\delta v^b = -\operatorname{div} v$ and $\Delta v^b = d\delta v^b + \delta dv^b$ is the Laplace–deRham operator on 1-forms.

Let v be the vector field $v_{ns} = f(t)(d\Phi)^\sharp$ on \mathbb{H}^2 as in *Theorem 2.1*. Because the 1-form $d\Phi$ is harmonic, one needs only to compute the covariant derivative term and the Ricci term:

$$\nabla_{v_{ns}} v_{ns}^b = \frac{1}{2} f^2(t) d|d\Phi|^2 \quad \text{and} \quad 2r(v_{ns}^b) = -2f(t)d\Phi.$$

Direct computation, taking into account the fact that for \mathbb{H}^2 we have $r = -1$, shows that both terms can be absorbed by the pressure term, so that the pair (v_{ns}^b, p) , where $p := (2f(t) - f'(t))\Phi - 1/2f^2(t)|d\Phi|^2$, satisfies Eq. 5.

Finally, a quick inspection shows that any differentiable function $f(t)$ satisfying

$$f^2(t) + 4 \int_0^t f^2(s) ds \leq f^2(0)$$

yields a vector field v_{ns} that satisfies the remaining conditions in Eq. 4 required of a Leray–Hopf solution.

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