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A NOTE ON THE ABUNDANCE OF DIFFERENTIAL COMBINANTS IN A FUNDAMENTAL SYSTEM

By Oliver E. Glenn

Department of Mathematics, University of Pennsylvania

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I. In a paper read before the section of algebra of the International Mathematical Congress at Toronto the present writer gave the definitions of the differential combinants of a set of p differential quantics,

$$f^{(r)} = a_{dx}^{(r)m} = (a_1^{(r)}\, dx_1 + \ldots + a_n^{(r)}\, dx_n)^m = (da^{(r)})^m,\ (r = 1, \ldots, p), \quad (1)$$

$p \geqq n$, and developed general methods of generating these parameters. The special property of combinant differential parameters is their invariance when the forms themselves are subject to linear transformations with constant coefficients.[1] No theory of complete systems is yet known even in the binary case. In fact the problem of the fundamental system of algebraic[2] combinants is far from completion and one of the most recent writers on the latter[3] subject confined his discussion, in lieu of such theory, to statistical data, giving tables of important combinants of low orders for binary forms of various orders without emphasis upon finite systems. The significance of such data results from the importance of combinants in applications[4] of invariant theory to geometry.

The object of the present article is to indicate the methods which are available for the construction of corresponding tables of differential combinants. In particular it is shown that the number of combinant parameters in a complete system is necessarily large.

II. Suppose there are two binary ground-forms, f, g, as in (1),

$$f = (d\alpha)^m = \alpha_{dx}^m, \qquad g = (d\beta)^m = \beta_{dx}^m = \beta_{dx}'^m. \quad (2)$$

We construct a set, S, of algebraic combinants of f, g, a fundamental system if one is known for the given order m. These, and all combinants found, are complemented by multiplication by the proper powers of the resultant of f and g to make all invariants[5] absolute. We then apply to

S the following operations which generate combinant parameters from combinants:

(*i*) The jacobian of f and g or of any pair of combinants.

(*ii*) The total differential of any combinant.

(*iii*) The operation δ and its iterations:

$$\delta = \zeta^{(1)} \frac{\partial}{\partial x_1} + \zeta^{(2)} \frac{\partial}{\partial x_2}, \tag{3}$$

in which $\zeta^{(1)}, \zeta^{(2)}$ are any elements cogredient to dx_1, dx_2.

(*iv*) Polarization by

$$P = \zeta^{(1)} \frac{\partial}{\partial(dx_1)} + \zeta^{(2)} \frac{\partial}{\partial(dx_2)}. \tag{4}$$

Having generated all possible combinants of orders $\leqq t$, by (*i*), . . ., (*iv*), we may form the algebraic simultaneous concomitant system of the totality and all parameters generated thus will be combinants of orders t or less.

The forms may be expressed either in Maschke's symbolic notation[5] or as actual functions.

III. Consider the problem of two quantics of the second order,

$$\begin{aligned} f &= (d\alpha)^2 = a_0(x_1, x_2)dx_1^2 + 2a_1(x_1, x_2)dx_1dx_2 + a_2(x_1, x_2)dx_2^2, \\ g &= (d\beta)^2 = b_0(x_1, x_2)dx_1^2 + 2b_1(x_1, x_2)dx_1dx_2 + b_2(x_1, x_2)dx_2^2. \end{aligned} \tag{5}$$

A complete system of algebraic combinants consists[6] of the first transvectant, J, of f and g and its discriminant D.

A combinant of order t of two m-ics is rational and integral in the determinants of order two of the matrix

$$\| a_{ix_1^{u-v}x_2^v},\ b_{ix_1^{u-v}x_2^v} \|, \quad (v = 0, \ldots, u;\ u = 0, \ldots, t;\ i = 0, \ldots, m), \tag{6}$$

where

$$a_{ix_1^{u-v}x_2^v} = \frac{\partial^u a_i}{\partial x_1^{u-v}\partial x_2^v}.$$

Each different set $(i, u-v, v)$ furnishes a row of the matrix. If we use the abbreviation

$$\begin{pmatrix} i & p & q \\ j & r & s \end{pmatrix} = a_{ix_1^p x_2^q}\, b_{jx_1^r x_2^s} - a_{jx_1^r x_2^s}\, b_{ix_1^p x_2^q}, \tag{7}$$

we have,

$$\begin{aligned} \frac{\partial}{\partial x_1}\begin{pmatrix} i & p & q \\ j & r & s \end{pmatrix} &= \begin{pmatrix} i & p+1 & q \\ j & r & s \end{pmatrix} + \begin{pmatrix} i & p & q \\ j & r+1 & s \end{pmatrix}, \\ \frac{\partial}{\partial x_2}\begin{pmatrix} i & p & q \\ j & r & s \end{pmatrix} &= \begin{pmatrix} i & p & q+1 \\ j & r & s \end{pmatrix} + \begin{pmatrix} i & p & q \\ j & r & s+1 \end{pmatrix}. \end{aligned} \tag{8}$$

The following is the complete list of combinants of orders $\leqq 1$, when[7]

$$\zeta^{(1)} = D^{-\frac{1}{4}}\frac{\partial u}{\partial x_2},\quad \zeta^{(2)} = -D^{-\frac{1}{4}}\frac{\partial u}{\partial x_1},\quad m = 2, \qquad (9)$$

and u is an arbitrary function of x_1, x_2. Note that subscripts of symbols indicate partial derivatives. Thus,

$$\alpha_{1dx} = \frac{\partial^2\alpha}{\partial x_1^2}\,dx_1 + \frac{\partial^2\alpha}{\partial x_1\partial x_2}\,dx_2 = \alpha_{11}\,dx_1 + \alpha_{12}\,dx_2. \qquad (10)$$

Symbolic forms

(1) $J = (\alpha\beta)\alpha_{dx}\beta_{dx}$, (2) $D = \frac{1}{2}(\alpha\beta)(\alpha'\beta')\ [(\alpha\beta')(\beta\alpha') + (\alpha\alpha')(\beta\beta')]$,

(3) $\frac{1}{4}(f, g) = (\alpha_{dx}\beta_{dx})\alpha_{dx}\beta_{dx}$, (4) $dD^{-\frac{1}{4}}J = \left(\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2\right)D^{-\frac{1}{4}}J$,

(5) $\delta D^{-\frac{1}{4}}J = (D^{-\frac{1}{4}}J, u)D^{-\frac{1}{4}}$

(6) Polars of the above forms by the operator P.

Leading coefficients

$$(1)\ \binom{000}{100};\quad (2)\ \binom{000}{200}^2 - 4\binom{000}{100}\binom{100}{200};\quad (3)\ \binom{010}{001},$$

$$(4)\ \left\{\binom{000}{100}\left[-\tfrac{1}{2}\binom{010}{200}\binom{000}{200} - \tfrac{1}{2}\binom{000}{210}\binom{000}{200} + \binom{010}{100}\binom{100}{200}\right.\right.$$
$$\left. + \binom{000}{110}\binom{100}{200} + \binom{000}{100}\binom{110}{200} + \binom{000}{100}\binom{100}{210}\right] + D\left[\binom{010}{100}\right.$$
$$\left.\left. + \binom{000}{110}\right]\right\}D^{-\frac{5}{4}} = \zeta_1 D^{-\frac{5}{4}},\quad (5)\ (\zeta u)D^{-\frac{3}{2}};\ \zeta_2 = \left[-\tfrac{1}{2}\binom{000}{200}\binom{001}{200}\right.$$
$$-\tfrac{1}{2}\binom{000}{200}\binom{000}{201} + \binom{001}{100}\binom{100}{200} + \binom{000}{101}\binom{100}{200} + \binom{000}{100}\binom{101}{200}$$
$$\left. + \binom{000}{100}\binom{100}{201}\right]\binom{000}{100} + D\left[\binom{001}{100} + \binom{000}{101}\right].$$

IV. The paper mentioned in paragraph I is mostly theoretical, presenting the theory of differential combinants of general order of p forms in n variables. It deals, also, with a principle of translation analogous to that of Meyer[8] for rational curves. The generalization of (6), III, is fundamental: Combinants of order r of the set (1) are functions of determinants of order p of the matrix

$$\left\|\, a^{(1)}_{i\ldots l}\,x_1^{s_1}\cdots x_n^{s_n},\ \ldots\ldots\ldots,\ a^{(p)}_{i\ldots l}\,x_1^{s_1}\cdots x_n^{s_n}\,\right\|,$$

$(s_1 + \ldots + s_n = t;\ t = 0, \ldots, r;\ i + j + \ldots + l = m)$, in which the letters $a^{(r)}_{i\ldots l}$ are the actual coefficients of $f^{(r)}$.

[1] Sylvester, *Camb. and Dublin Math. J.*, **8**, 1853, 62–64.
[2] Gordan, *Math. Ann.*, **5**, 1872, 95–122.
[3] Shenton, *Amer. J. Math.*, **37**, 1915, 247–271.
[4] Rowe, *Trans. Am. Math. Soc.*, **12**, 1911, 295–310.
[5] Maschke, *Trans. Amer. Math. Soc.*, **4**, 1903, 446, 448.
[6] Glenn, *Theory of Invariants*, 1915, 167.
[7] *Encyc. der math. Wiss.*, **1** (IB2), 385.
[8] Meyer, *Apolarität und Rat. Curv.*, 1883, 18.

ON THE APPLICATION OF BOREL'S METHOD TO THE SUMMATION OF FOURIER'S SERIES[1]

BY CHARLES N. MOORE

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CINCINNATI

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There are two ways of studying the relationship between various methods for the summation of divergent series. One consists in the attempt to determine directly whether or not one is more general than the other, and if this is not the case to determine under what conditions both methods apply and give the same sum to the series that is used. Another method involves the determination of the relative scope of the various processes in summing certain general types of series that are of fundamental importance in analysis. The former method is more exhaustive from the theoretical point of view; the latter is, perhaps, of greater practical interest.

The two most important types of series in analysis at the present time are power series and Fourier's series. It is well known that Cesàro's method will not sum a power series outside of its circle of convergence, whereas Borel's method applies everywhere within the polygon of summability. However, in the case where the circle of convergence is a natural boundary, Cesàro's method may be applicable at points on the circle of convergence where Borel's method fails. This phase of the relationship between the two methods may well be described by an illuminating remark made by G. H. Hardy[2] in another connection, namely, that "Borel's method, although more powerful than Cesàro's, is never more delicate, and often less so."

Cesàro's method has been found to be admirably adapted to the study of Fourier's series. It will give the proper sum for the Fourier's series of any continuous function at all points, and will sum the Fourier's series of any function having a Lebesgue integral to the value of the function, except perhaps at a set of points of measure zero. Since there is considerable similarity in the behavior of Fourier's series and the behavior of power series on the circle of convergence, it is natural to expect that Borel's