COMMENTARY

Model theory and the cardinal numbers $\mathfrak{p}$ and $\mathfrak{t}$

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Modern mathematical logic is a multifaceted subject, which concerns itself with the strengths and limitations of formal proofs and algorithms and the relationship between language and mathematical structure. Modern mathematical logic also addresses foundational issues that arise in mathematics. This commentary summarizes the groundbreaking results of Malliaris and Shelah (1), recently published in PNAS (2), relating two branches of logic: model theory and set theory.

Higher Orders of Infinity

At the end of the 19th century, Cantor made the remarkable discovery that it was possible to develop theory of the size or cardinality of an infinite set. Two sets, X and Y, have the same cardinality if there is a bijective correspondence between them: that is, there is a pairing between the elements of X and Y so that each element of one corresponds uniquely to an element of the other. Sets that are either finite or have the same cardinality as the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ are said to be countable; otherwise, they are uncountable. Cantor demonstrated that the rational numbers are countable but the real numbers are uncountable: their cardinalities are commonly denoted $\aleph_0$ and $2^{\aleph_0}$, respectively. The distinction between the countable and the uncountable is very important in mathematics. For example, the existence of a countable set of points in a manifold or Hilbert space, which can be used to arbitrarily approximate all other points, is crucially used in many places throughout mathematical analysis.

Set theory—one of the four main branches of modern logic—concerns itself with foundational issues relating to uncountable sets. One of the earliest, and surely the most famous, problems in set theory asked whether there was a cardinal number that was strictly between $\aleph_0$ and $2^{\aleph_0}$; the assertion that no such intermediate cardinal exists is known as the "continuum hypothesis." In complementary results, Gödel (3) and Cohen (4, 5) demonstrated that the continuum hypothesis is neither provable nor refutable based on the standard axioms of mathematics.

Just as new models of geometry were discovered and explored by Bolyani, Gauss, and Lobachevsky in the 19th century, modern set theory concerns itself with building and understanding models of the axioms of mathematics. If the continuum hypothesis fails, then there are orders of infinity that lie somewhere between $\aleph_1$—the smallest uncountable cardinal—and $2^{\aleph_0}$. Two examples are $\mathfrak{p}$ and $\mathfrak{t}$. The cardinal $\mathfrak{p}$ is the minimum cardinality of a collection F of infinite subsets of $\mathbb{N}$, all of whose finite intersections are infinite, such that there is no single infinite $A \subseteq \mathbb{N}$, such that every element of F contains A except for a finite error. The cardinal $\mathfrak{t}$ is defined similarly, except one only quantifies over families F which are totally ordered by containment modulo a finite error. Although $\mathfrak{p} \leq \mathfrak{t}$ is immediate from the definitions, it has been an open problem for over 50 years whether $\mathfrak{p} = \mathfrak{t}$ is provable from the axioms of mathematics or whether, like the continuum hypothesis, it is undecidable based on the axioms. These cardinals, and $\mathfrak{p}$ in particular, arise in the study of more exotic topological spaces in topology and analysis. For example, $\mathfrak{p}$ is the cardinality of the smallest number of nowhere dense subsets which are needed to cover a separable compact topological space.

Malliaris and Shelah outline a proof that $\mathfrak{p} = \mathfrak{t}$—a very rare instance of a provable equality among orders of infinity.

Curiously, it has long been known that if $\mathfrak{p} = \aleph_1$, the smallest possible value it can take, then $\mathfrak{p} = \mathfrak{t}$. Thus, if $\mathfrak{p} < \mathfrak{t}$ is true in a model of mathematics, then necessarily $2^{\aleph_0}$ is at least $\aleph_3$, the third uncountable cardinal. This is interesting because we have a much more limited understanding of models of mathematics in which $2^{\aleph_0} > \aleph_2$ than those in which $2^{\aleph_0} \leq \aleph_2$. Still, it was widely believed that $\mathfrak{p} < \mathfrak{t}$ should consistent with the axioms of mathematics.

Model Theory and Classification

Model theory is a different branch of mathematical logic that concerns itself with the semantics and syntactics of abstract mathematical structures, such as the field of complex numbers $(\mathbb{C}, +, \cdot)$ and linear orders such as $(\mathbb{Q}, <)$. One of the results that modernized the subject was Morley's categoricity theorem (6). A fundamental result in linear algebra is that any two vector spaces that have the same dimension and same field of scalars are isomorphic. If the field of scalars is the rational numbers, then the dimension of an uncountable vector space is the same as its cardinality. Thus, any two vector spaces over $\mathbb{Q}$, which have the same uncountable cardinality, are in fact isomorphic. The categoricity theorem is a vast generalization of this; it asserts that, for any theory T and uncountable cardinals κ and λ, if any two models of T of cardinality κ are isomorphic, then any two models of cardinality λ are isomorphic. The proof proceeds by showing that, even in this generality, theories that satisfy the hypothesis of the theorem admit an abstract notion of a basis, much as in the setting of vector spaces.

This rather extreme characteristic of a theory is by no means typical. For example, both $\mathbb{R}$ and $\mathbb{Q} \cup (0, 1)$ are dense linear orders of the same cardinality, but they are not isomorphic: every interval in $\mathbb{R}$ is uncountable, whereas the interval of points between 2 and 3 in $\mathbb{Q} \cup (0, 1)$ is countable. Theories that are close to that of the algebraic structure $(\mathbb{C}, +, \cdot)$ are generally regarded in model theory as being tame, whereas those similar to the dense linear order $(\mathbb{R}, <)$ are regarded as being wild. In the 1970s, Shelah initiated his classification program in model theory in an attempt to separate the tame from the wild and to stratify what lay in between (7).

One method of stratifying theories is Keisler's order (8), which measures how easily the models of the theory become saturated by taking ultrapowers. The ultrapower construction is a useful tool by which a

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structure is enlarged while preserving its theory. The logical properties of the ultrapower are obtained by integration via a certain binary valued measure, known as an ultrafilter. Such enlargements have a greater tendency to contain solutions to large systems of logical equations. Those structures that have solutions to any consistent system of logical equations, which are smaller than the structure itself, are said to be saturated. The tamer the theory, the more likely its ultrapowers are to be saturated.

It was long known that the theories of $(\mathbb{C}, +, \cdot)$ and $(\mathbb{R},<)$ provided examples at the opposite ends of this spectrum, and that the presence of a definable ordering in the structure was an indication of high complexity. Shelah also characterized, in model theoretic terms, the smallest two classes in Keisler's order: these classes together are the "stable theories" (4). No model theoretic characterization was known, however, of the maximal theories in Keisler's order. Over the decades that followed, very little progress was made in this direction.

Recent Developments

In a watershed moment in both model theory and set theory, Malliaris and Shelah solved both of these central—and seemingly unrelated problems—by realizing they are in fact closely connected. In their article in PNAS (2), Malliaris and Shelah outline a proof that $\mathfrak{p} = \mathfrak{t}$—a very rare instance of a provable equality among orders of infinity—by a tour de force analysis of definability in ultraproducts of finite linear orders. Using this same analysis, Malliaris and Shelah show that even the weak presence of a definable linear ordering in the structure—specifically Shelah's property SOP_3—exactly characterize the maximal class of theories in Keisler's order. This finding suggests a wealth of theories of intermediate complexity. The analysis is also unique in proving that there are theories more complex than the stable theories but less complex than the maximal class in Keisler's order, and opens the door for new interactions between set theory and model theory.

1 Malliaris M, Shelah S (2012) Cofinality spectrum problems in model theory, set theory, and general topology. arxiv:math.LO/1208.5424.
2 Malliaris M, Shelah S (2013) General topology meets model theory, on $\mathfrak{p}$ and $\mathfrak{t}$. *Proc Natl Acad Sci USA* 110:13300–13305.
3 Gödel K (1940) The consistency of the continuum hypothesis. *Annals of Mathematics Studies, no. 3* (Princeton Univ Press, Princeton, NJ).
4 Cohen PJ (1963) The independence of the continuum hypothesis. *Proc Natl Acad Sci USA* 50(6):1143–1148.
5 Cohen PJ (1964) The independence of the continuum hypothesis. II. *Proc Natl Acad Sci USA* 51(1):105–110.
6 Morley M (1965) Categoricity in power. *Trans Am Math Soc* 114:514–538.
7 Shelah S (1978) Classification theory and the number of nonisomorphic models. *Studies in Logic and the Foundations of Mathematics*, vol 92. (North-Holland Publishing, Amsterdam, The Netherlands).
8 Keisler J (1967) Ultraproducts which are not saturated. *J Symbolic Logic* 32:23–46.