Structure of Leavitt path algebras of polynomial growth

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We determine the structure of Leavitt path algebras of polynomial growth and discuss their automorphisms and involutions.

Gelfand–Kirillov dimension | Toeplitz algebra

Following the works in refs. 1–4, Abrams and Aranda Pino (5) and Ara et al. (6) introduced Leavitt path algebras of directed graphs as algebraic analogs of C* algebras of Cuntz and Krieger. This construction provided a rich supply of finitely presented algebras having interesting and extreme properties.

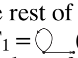
Let $\Gamma=(V,E)$ be a finite directed graph with the set of vertices V and the set of edges E. For an edge $e\in E$, we let $s(e)$ and $r(e)\in V$ denote its source and range, respectively. A vertex v for which $s^{-1}(v)$ is empty is called a *sink*. A *path* $p=e_1\dots e_n$ in a graph Γ is a sequence of edges $e_1\dots e_n$ such that $r(e_i)=s(e_{i+1})$, $i=1,2,\dots,n-1$. In this case we say that the path p starts at the vertex $s(e_1)$ and ends at the vertex $r(e_n)$. If $s(e_1)=r(e_n)$ then the path is closed. If $p=e_1\dots e_n$ is a closed path and the vertices $s(e_1),\dots,s(e_n)$ are distinct, then the subgraph $(\{s(e_1),\dots,s(e_n)\},\{e_1,\dots,e_n\})$ of the graph Γ is called a cycle.

Let Γ be a finite graph and let F be a field. The Leavitt path F algebra $L(\Gamma)$ is the F algebra presented by the set of generators $\{v|v\in V\}$ $\{e,e^*|e\in E\}$ and the set of relations (i) $v_iv_j=\delta_{v_i,v_j}v_i$ for all $v_i,v_j\in V$; (ii) $s(e)e=er(e)=e, r(e)e^*=e^*s(e)=e^*$ for all $e\in E$; (iii) $e^*f=\delta_{e,f}\,r(e)$ for all $e,f\in E$; and (iv) $v=\sum_{s(e)=v}ee^*$ for an arbitrary vertex $v\in V\backslash\{\text{sinks}\}$. The mapping that sends v to v, $v\in V$, e to e^*, and e^* to e, $e\in E$, extends to an involution of the algebra $L(\Gamma)$. If $p=e_1\dots e_n$ is a path, then $p^*=e_n^*\dots e_1^*$.

In ref. 7 we showed that the algebra $L(\Gamma)$ has polynomial growth if and only if no two cycles of Γ intersect. Let $N=\{1,2,\dots\}$, and let $n\in N$. For an algebra R, let $M_n(R)$ denote the algebra of $n\times n$ matrices over R and let $M_\infty(R)$ denote the algebra of infinite $N\times N$ finitary matrices over R, that is, infinite $N\times N$ matrices with only finitely many nonzero entries.

Theorem 1. *Let $L(\Gamma)$ be a Leavitt path algebra of polynomial growth. Then $L(\Gamma)$ has a finite chain of ideals, $(0)\le I_0<I_1<\dots<I_s=L(\Gamma)$, such that I_0 is a finite sum of matrix algebras and infinite finitary matrix algebras over F and each factor I_{i+1}/I_i, $i\ge 1$, is a finite sum of matrix algebras and finitary matrix algebras over the Laurent polynomial algebra $F[t^{-1},t]$. The ideals I_i are invariant under Aut $(L(\Gamma))$.*

Remark 1: We will show that I_0 is the locally finite radical of $L(\Gamma)$ (8).

In the rest of the paper we study the algebraic Toeplitz algebra $L(\Gamma_1),\Gamma_1=$  (9) as the simplest nontrivial example of a Leavitt path algebra of polynomial growth. As shown in ref. 9 (it follows also from *Theorem 1* above) the locally finite radical I_0 of $L(\Gamma_1)$ is $M_\infty(F)$ and $L(\Gamma_1)/M_\infty(F)\cong F[t^{-1},t]$.

Theorem 2. *The short exact sequence $(0)\to M_\infty(F)\to L(\Gamma_1)\to F[t^{-1},t]\to(0)$ does not split.*

The significance of *Theorem 2* is that it shows that the extensions in *Theorem 1*, generally speaking, do not split.

We describe automorphisms and involutions of the algebraic Toeplitz algebra $L(\Gamma_1)$. Description of involutions is related to the question of whether isomorphic Leavitt path algebras are isomorphic as involutive algebras (10).

Theorem 3. $\mathrm{Aut}(L(\Gamma_1))\cong F^*\ltimes GL_\infty(F)$, *a semidirect product of the multiplicative group F^* of the field F with the general linear finitary group $GL_\infty(F)$. If $F^2=F$ then the only involution on $L(\Gamma_1)$ (up to isomorphism) is the standard involution $*$.*

In what follows we will assume that the finite graph Γ does not have distinct intersecting cycles, which guarantees that $L(\Gamma)$ has polynomial growth. For an arbitrary path p, the element pp^* is an idempotent. Consider the family of idempotents $\tilde{\mathcal{E}}=\{pp^*|p \text{ is a path}\}$.

Remark 2: We view vertices as paths of length 0.

For two idempotents $e=pp^*$, $f=qq^*\in\tilde{\mathcal{E}}$, if neither p nor q is an initial subpath of the other, then e and f are orthogonal. If $p=qp'$ then $ef=fe=e$.

Consider the set of vertices $V_0=\{v\in V|$ no path starting at v finishes at a cycle$\}$. The subset V_0 is hereditary and saturated (5). Hence, the ideal $I_0=\mathrm{id}_{L(\Gamma)}(V_0)$ is the F span of all products pq^*, where p, q are paths, $r(p)=r(q)\in V_0$. Let $\mathcal{E}=\{pp^*\in\tilde{\mathcal{E}}|r(p)\in V_0\}\subseteq\tilde{\mathcal{E}}$. Because $pq^*=(pp^*)(pq^*)(qq^*)$ it follows that $I_0=\mathcal{E}I_0\mathcal{E}$. Consider also the set of idempotents $\mathcal{E}_S=\{pp^*\in\mathcal{E}|r(p)$ is a sink$\}$. We call idempotents from $\mathcal{E}_S$ minimal. Let $v_1,\dots,v_l$ be all sinks of Γ. Let $\mathcal{E}_i=\{pp^*\in\mathcal{E}|r(p)=v_i\}$. Clearly, $\mathcal{E}_S=\mathcal{E}_1\dot\cup\cdots\dot\cup\mathcal{E}_l$, and $\mathcal{E}_i\mathcal{E}_j=(0)$ if $i\ne j$.

Lemma 4 (6).

i) Every idempotent from $\mathcal{E}$ is a sum of minimal idempotents,

ii) if $e\in\mathcal{E}_S$ then $eL(\Gamma)e=Fe$,

iii) if $e\in\mathcal{E}_i, f\in\mathcal{E}_j$ then $\dim_F eL(\Gamma)f=\delta_{ij}$.

The set $\mathcal{E}_i$ is infinite if and only if there exists a cycle from which one can get to v_i. In that case *Lemma 4* implies that $\mathcal{E}_iL(\Gamma)\mathcal{E}_i\cong M_\infty(F)$. Otherwise $\mathcal{E}_iL(\Gamma)\mathcal{E}_i\cong M_k(F)$, where k is the number of paths that end at v_i. We thus have proved that I_0 is isomorphic to a finite sum of matrix algebras and infinite finitary matrix algebras over F.

Recall that an algebra is said to be *locally finite dimensional* if every finitely generated subalgebra of it is finite dimensional. The sum of all locally finite-dimensional ideals of an associative algebra A is a *locally finite-dimensional ideal*, which is called the *locally finite-dimensional radical,* denoted by $\mathrm{Loc}(A)$. For further properties of $\mathrm{Loc}(A)$, see ref. 8.

Lemma 5. $I_0=Loc(L(\Gamma))$.

The ideal I_0 is also the socle of the algebra $L(\Gamma)$ (11).

As shown in ref. 5 $L(\Gamma)/I_0\cong L(\Gamma')$, where $\Gamma'=(V\backslash V_0,E\backslash r^{-1}(V_0))$; the graph Γ' does not have sinks. Without loss of generality consider therefore a finite graph Γ such that $GK_{\dim}L(\Gamma)<\infty$ and Γ does not have sinks, so $I_0=\mathrm{Loc}(L(\Gamma))=(0)$.

Recall that an edge e is called an exit from a cycle C if $s(e)$ lies on C, but e is not a part of C (5). A cycle without exits will be referred to as an *NE* cycle. For an arbitrary vertex $v\in V$ there exists a path that starts at v and ends on a cycle, otherwise $v\in V_0$,

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which contradicts our assumption. Moreover, because distinct cycles of Γ do not intersect and all chains of cycles (7) are finite, it follows that for an arbitrary $v \in V$ there exists a path that starts at v and ends on an *NE* cycle.

Consider the set $V_1 = \{v \in V|$ a path that starts at v can end only on an *NE* cycle$\}$. The set V_1 is obviously hereditary and saturated. Let $C_1, \ldots, C_l$ be all *NE* cycles of Γ, $\mathcal{E}_i' = \{pp^*|p$ is a path, $r(p) \in C_i\}$. Clearly, $\mathcal{E}_i'\mathcal{E}_j' = (0)$ if $i \neq j$. We define $J = \mathrm{id}_{L(\Gamma)}(V_1)$. Then J = $\mathrm{span}(pq^*|r(p) = r(q) \in V_1) = \oplus_{i=1}^{l} \mathcal{E}_i' J \mathcal{E}_i'$.

Consider an *NE* cycle C_i with d_i vertices. In ref. 12 it is shown that the subalgebra $L(C_i) = \mathrm{span}(pq^*|p,\ q$ are both paths on the cycle $C_i)$ is isomorphic to $M_{d_i}(F[t^{-1}, t])$.

Lemma 6. *Let $e = pp^*$, $f = qq^* \in \mathcal{E}_i'$. Then $eL(\Gamma)f = pL(C_i)q^*$.*

If the set $\mathcal{E}_i'$ is infinite, which happens if there exists a cycle C different from C_i and a path p such that $s(p) \in C$, $r(p) \in C_i$, then by *Lemma 6* $\mathcal{E}_i' L(\Gamma) \mathcal{E}_i' \cong M_\infty(F[t^{-1}, t])$. If $|\mathcal{E}_i| = k$ then $\mathcal{E}_i' L(\Gamma) \mathcal{E}_i' \cong M_k(F[t^{-1}, t])$.

We have proved that the algebra J is isomorphic to a finite direct sum of matrix algebras and infinite finitary matrix algebras over $F[t^{-1}, t]$. The ideal I_0 of the algebra $L(\Gamma)$ has been defined. For $i \geq 1$ define I_i via $I_i/I_{i-1} = J(L(\Gamma)/I_{i-1})$. We have got an ascending chain claimed in *Theorem 1*. The ideal $I_0 = \mathrm{Loc}\, L(\Gamma)$ is invariant under $\mathrm{Aut}\, L(\Gamma)$. To prove that the ideals $I_i, i \geq 1$, are invariant we need to obtain an abstract characterization of the ideal J.

Lemma 7. *The ideal J is the largest ideal of $L(\Gamma)$ with the property that for an arbitrary element $a \in J$, and an arbitrary finite-dimensional subspace G of $L(\Gamma)$ that generates $L(\Gamma)$, there exists a positive constant $k = k(a, G)$ such that $\dim_F(aG^n a) \leq kn$ for $n \geq 1$.*

Corollary 8. *Let Γ_1 and Γ_2 be finite graphs, and suppose that $\phi : L(\Gamma_1) \to L(\Gamma_2)$ is an isomorphism and that $L(\Gamma_1)$ has polynomial growth. Then $\phi(J(L(\Gamma_1))) = J(L(\Gamma_2))$.*

Theorem 1 is proved.

We determined the factors I_{i+1}/I_i, but the nature of extensions remains unclear. *Theorem 2* implies that generally speaking they do not split.

The algebra $L(\Gamma_1)$ can be presented by generators and relators as $A = \langle x, y | xy = 1\rangle$ (13). Let us fix the notation. The element $e = yx$ is an idempotent. We have

$$I = A(1-e)\,A = \sum_{i,j=1}^{\infty} Fe_{ij}, e_{ij} = y^{i-1}(1-e)x^{j-1}, e_{ij}e_{pq} = \delta_{jp}e_{iq}, x(1-e)$$
$$= (1-e)\,y = 0;$$
$$I \cong M_\infty(F),\ A/I \cong F\left[t^{-1}, t\right].$$

Suppose that the extension splits, that is, the algebra A contains a subalgebra B, which is isomorphic to $F[t^{-1}, t]$, $A = B + I$. Let $x = b_1 + \sum_{i,j} \alpha_{ij}e_{ij}$, $1 = b_0 + \sum_{i,j} \beta_{ij}e_{ij}, y = b_{-1} + \sum_{i,j} \gamma_{ij}e_{ij}$ where $b_{-1}, b_0,\ b_1 \in B$; $\alpha_{ij},\ \beta_{ij},\ \gamma_{ij} \in F$. Consider the finite sets $P(b_1) = \{i|\alpha_{ij} \neq 0$ for some $j\}$, $P(b_{-1}) = \{j|\gamma_{ij} \neq 0$ for some $i\}$ and one-sided ideal $\rho = \sum_{i \in P(b_1), r \geq 1} Fe_{ir} \triangleleft_r I$, $\sigma = \sum_{j \in P(b_{-1}), r \geq 1} Fe_{rj} \triangleleft_e I$.

Lemma 9. *For an arbitrary element $a \in I$ there exists $N(a) \geq 1$ such that $b_1^k a \in \rho$, $ab_{-1}^k \in \sigma$, for any $k \geq N(a)$.*

Proof: Choose an element $a = \sum_{i,j} \xi_{ij}e_{ij}, 0 \neq \xi_{ij} \in F$. We define $|a| = max\{i \notin P(b_1)|\xi_{ij} \neq 0\}$. Because $xe_{ij} = e_{i-1,j}$ for $i > 1$, $xe_{1j} = 0$, it follows that $|b_1 a| < |a|$ or $a \in \rho$. This implies the first inclusion. The second inclusion is proved in the same way. This completes the proof of the Lemma. ∎

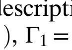
Lemma 10. *For an arbitrary element $a \in I$, we have $\dim_F aB < \infty$.*

Proof: Let $a \in I$. For arbitrary integers $p, q \geq N(a)$ we have $b_1^p a b_{-1}^q \in \rho \cap \sigma$. Notice that $\dim_F \rho \cap \sigma < \infty$. Fix $p \geq N(a)$. It follows from the above that there exists a nonzero polynomial $f(t) \in F[t]$ such that $b_1^p af(b_{-1}) = 0$. Because every nonzero ideal of the algebra $F[t^{-1}, t]$ is of finite codimension, we conclude that $\dim_F b_0 aB < \infty$. The element b_0 is the identity of the algebra B. Notice that $BI \neq (0)$. Otherwise IB is a nilpotent left ideal of the algebra I, which implies that $IB = (0)$, $A = B \oplus I$ is a direct sum. However, the algebra $F[t^{-1}, t] \oplus M_\infty(F)$ is not finitely generated, a contradiction. In view of the simplicity of the algebra I, the subset $b_0 I$ generates I as an ideal. This completes the proof of the Lemma.

Lemma 11. *For an arbitrary element $a \in I$, we have $\dim_F aA < \infty$.*

Proof: Let a_1 denote the sum $a_1 = \sum_{i,j} \alpha_{ij}e_{ij}$, and let a_{-1} denote the sum $a_{-1} = \sum_{i,j} \gamma_{ij}e_{ij}, x = b_1 + a_1$, and $y = b_{-1} + a_{-1}$. Let $a \in I$ and let $d = \max(\dim_F aB, \dim_F a_{-1}B, \dim_F a_1 B)$. We claim that efor each element $u \in \{a, a_{-1}, a_1\}$ and for an arbitrary product b of elements $b_1,\ b_{-1}$ of length $d+1$ we have $ub = \sum_k \xi_k ub^{(k)}$, where $\xi_k \in F, b^{(k)}$ are products of lements $b_1,\ b_{-1}$ of length $\leq d$. Indeed, consider the ascending chain of subspaces $Fu \subseteq uB^{(1)} \subseteq \cdots \subseteq uB^{(d+1)}$, where $B^{(k)}$ is the F span of all products of elements b_1, b_{-1} of length $\leq k$. Because $\dim_F uB \leq d$ we cannot have a strict inclusion at every step. Hence $uB^{(d)} = uB^{(d+1)}$, as claimed. Every product of elements $x,\ y$ is a linear combination of products of $b_1,\ b_{-1},\ a_1,\ a_{-1}$. Let w be a product of elements $b_1,\ b_{-1},\ a_1,\ a_{-1}$. Then aw can be represented as $aw = v_1 w_1 v_2 \ldots v_s w_s$, where v_i are products of $a,\ a_{-1},\ a_1$; w_i are products of $b_1,\ b_{-1}$. Because of the presence of the element a at the left end the word v_1 is not empty. The claim above implies that the words $w_1, \ldots, w_s$ can be assumed to have lengths $\leq d$. Now aw lies in the subalgebra of $M_\infty(F)$ generated by $ab,\ a_{-1}b$, $a_1 b$, where elements b are products in $b_1,\ b_{-1}$ of lengths $\leq d$. This subalgebra is finitely generated, hence finite dimensional. This completes the proof of the Lemma. ∎

It is well known that the set $\{x^i y^j : i, j \geq 0\}$ is a basis of A. Hence the elements $(1 - yx)y^i$, $i \geq 0$, are linearly independent, $\dim_F(1-e)A = \infty$, a contradiction. *Theorem 2* is proved.

Now our aim is description of automorphisms and involutions of the algebra $L(\Gamma_1)$, $\Gamma_1 =$ .

Consider the countably infinite-dimensional vector space $V = \sum_{i=1}^{\infty} Fe_i$. Let E be the algebra of all linear transformations of V. Because the basis $\{e_i, i \geq 1\}$ has been fixed we can identify E with the algebra of $N \times N$ matrices having only finitely many nonzero entries in each column. Consider also the subalgebra E_0 of E which consists of $N \times N$ matrices having finitely many nonzero entries in each row and in each column. As above, $M_\infty(F)$ is the algebra of finitary (having finitely many nonzero entries) $N \times N$ matrices. It is easy to see that $M_\infty(F)$ is an ideal in E_0 and a left ideal in E.

As follows from *Theorem 1*, the ideal $I_0 = \mathrm{id}_{L(\Gamma_1)}(v_2)$ is isomorphic to $M_\infty(F)$. Extending this isomorphism we can embed $L(\Gamma_1)$ into the algebra E_0, the cycle c and its conjugate c^* are identified with the matrices $c = \sum_{i=1}^{\infty} e_{i+1,i}$, $c^* = \sum_{i=1}^{\infty} e_{i,i+1}$, respectively, e_{ij} are matrix units, $L(\Gamma_1) = \, <c, c^*, M_\infty(F)>$.

Theorem 12. *(Jacobson, ref. 14). For an arbitrary automorphism φ of $M_\infty(F)$ there exists an invertible element $T \in E$ such that $\varphi(a) = T^{-1}aT$ for any $a \in M_\infty(F)$.*

Lemma 13. *An automorphism of $L(\Gamma_1)$ induces an automorphism of the type $t \to \alpha t$, $0 \neq \alpha \in F$, $L(\Gamma_1)/I_0 \cong F[t^{-1}, t]$.*

Proof: If the assertion is not true then there exists an automorphism φ of $L(\Gamma_1)$ whose image in $\mathrm{Aut}\, F[t^{-1}, t]$ maps t to t^{-1}. By Jacobson's theorem there exists an invertible element $T \in E$ such that $T^{-1}aT = \varphi(a)$ for all $a \in L(\Gamma_1)$. In particular, $T^{-1}cT = c^* + a$, $a \in M_\infty(F)$. Hence, $cT = Tc^* + Ta$, $Ta \in M_\infty(F)$. This implies that for a sufficiently large $n_0 \geq 1$ we have $(cT)_{ij} = (Tc^*)_{ij}$ provided that $i + j \geq n_0$. Therefore, $T_{i-1,j} = T_{i,j-1}$. We showed that $T_{ij} = \alpha_{i+j} \in F$ for $i + j \geq n_0$. The jth column of the matrix T intersects all diagonals $\{(i,j)|i + j = k\}$, $k \geq j$. Hence if the sequence α_k, $k \geq 1$, contains infinitely many nonzero entries then every column of T contains infinitely many nonzero entries. Hence the matrix T is finitary, a contradiction. This completes the proof of the Lemma. ∎

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Lemma 14. *If $T \in E$ is invertible and $T^{-1}M_\infty(F)T = M_\infty(F)$ then $T \in E_0$.*

Recall that the group $GL_\infty(F)$ of invertible matrices from $Id + M_\infty(F)$ is called the finitary general linear group (15). It can be realized as the union $GL_\infty(F) = \bigcup_{n\geq 1} GL_n(F)$.

Lemma 15. *Let $\varphi \in Aut\, L(\Gamma_1)$, $\varphi|_{L(\Gamma_1)/I_0} = Id$, $T \in E$, $\varphi(a) = T^{-1}aT$ for any $a \in L(\Gamma_1)$. Then $T = \alpha \cdot Id + a$, $0 \neq \alpha \in F$, $a \in M_\infty(F)$.*

Proof: By our assumptions $T^{-1}cT = c + a$, $a \in M_\infty(F)$, or, equivalently, $cT = Tc + Ta$, $Ta \in M_\infty(F)$. Hence for a sufficiently large $n_0 \geq 1$ $(cT)_{ij} = (Tc)_{ij}$, $T_{i+1,j} = T_{i,j-1}$ provided that $i + j \geq n_0$ (we assume that $T_{i,0} = 0$). Hence T is an almost Toeplitz matrix, $T = T_0 + \sum_{u\in Z} \alpha_k c^{(k)}$, where $T_0 \in M_\infty(F)$, $c^{(k)} = \sum_{j-i=k} e_{ij}$, $\alpha_k \in F$. The jth column intersects all diagonals $\{(i,j) | j - i = k\}$ with $k \leq j$. Hence the set $\{k < 0 | \alpha_k \neq 0\}$ is finite. Similarly, an ith row intersects all diagonals $\{(i,j) | j - i = k\}$ with $-k \leq i$. Hence the set $\{k > 0 | \alpha_k \neq 0\}$ is finite as well. Now we have $T = T_0 + \sum_{k=-m}^{n} \alpha_k c^{(k)}$; $\alpha_{-m} \cdot \alpha_n \neq 0$. Because the matrix $\sum_{n=-m}^{n} \alpha_k c^{(k)}$ cannot be strictly upper or lower triangular (otherwise T would not be invertible), we can assume that $m, n \geq 0$. All of the above applies to the matrix T^{-1} as well, $T^{-1} = (T^{-1})_0 + \sum_{s=-p}^{q} \beta_s c^{(s)}$; $\beta_q \cdot \beta_{-p} \neq 0$; $p, q \geq 0$, $(T^{-1})_0$ is a finitary matrix. Now

$$Id = \left(T_0 + \sum_{k=-m}^{n} \alpha_k c^{(k)}\right)\left(\left(T^{-1}\right)_0 + \sum_{s=-p}^{q} \beta_s c^{(s)}\right)$$
$$= T_0 \cdot T^{-1} + (T - T_0)\left(T^{-1}\right)_0 + \sum \alpha_i \beta_j c^{(i+j)}.$$

Because T, $T^{-1} \in E_0$ it follows that $T_0 \cdot T^{-1} + (T - T_0)(T^{-1})_0 \in M_\infty(F)$. Moreover, the equality above implies that $m = n = p = q = 0$, $T = \alpha_0 \cdot Id + T_0$. This completes the proof of the Lemma, and thus completes the proof of *Theorem 3*. ∎

Consider the embedding $\pi : F^* \to E_0^*$ of the multiplicative group of the field F into the multiplicative group of the algebra E_0, $\pi(\alpha) = \mathrm{diag}(1, \alpha, \alpha^2, \ldots)$. It is easy to see that $\pi(\alpha)^{-1}L(\Gamma_1)\pi(\alpha) = L(\Gamma_1)$ and $\pi(\alpha)^{-1}c\pi(\alpha) = \alpha c$. Now, *Lemmas 13* and *15* imply that $\mathrm{Aut}(L(\Gamma_1)) = \pi(F^*) \ltimes GL_\infty(F)$.

We say that two involutive algebras $(R_1, *_1)$ and $(R_2, *_2)$ are isomorphic if there exists an isomorphism $\varphi : R_1 \to R_2$ of algebras R^1, R_2, such that $\varphi(a^{*_1}) = \varphi(a)^{*_2}$ for an arbitrary element $a \in R_1$.

Lemma 16. *Let $F^2 = F$. Then the algebra $L(\Gamma_1)$ has only one (up to isomorphism) involution: the standard involution $*$.*

Proof: If we view $L(\Gamma_1)$ as a subalgebra of the algebra E_0, then the standard involution $*$ becomes the restriction of the transposition $(a_{ij})^t = (a_{ji})$. Let $: L(\Gamma_1) \to L(\Gamma_1)$ be an involution. The composition of the involutions $-$ and t is an automorphism. Hence there exists a matrix $T \in \pi(F^*)GL_\infty(F)$ such that $(\overline{a})^t = T^{-1}aT$ for all elements $a \in L(\Gamma_1)$, $\overline{a} = T^t a^t (T^t)^{-1}$. Applying the involution $-$ twice we get $a = \overline{\overline{a}} = T^t(T^{-1}aT)(T^t)^{-1} = (T^tT^{-1})a(T(T^t)^{-1})$. Because the matrix $(T^t)T^{-1}$ commutes with an arbitrary matrix from $M_\infty(F)$ it follows that $T^tT^{-1} = \alpha \cdot Id$, $\alpha \in F^*$, $T^t = \alpha T$. Now, $T = (T^t)^t = \alpha^2 T$, $\alpha = \pm 1$. All nonzero entries of the matrix T except finitely many lie in the main diagonal. Hence T cannot be skew-symmetric. Hence $T^t = T$. If an arbitrary element from F is a square then there exists a matrix $Q \in \pi(F^*)$ $GL_\infty(F)$ such that $T = Q^tQ$. Now the mapping $a \to Q^{-1}aQ$ is an isomorphism of the involutive algebra $(L(\Gamma_1), t)$ to the involutive algebra $(L(\Gamma_1), *)$. This completes the proof of the Lemma. ∎

Note Added in Proof. For a different approach to automorphisms of the Jacobson algebra, see ref. 16.

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