SPECIAL FEATURE

Positive basis for surface skein algebras

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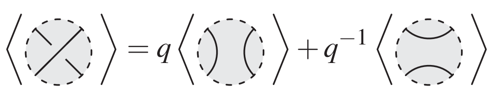
We show that the twisted SL_2 skein algebra of a surface has a natural basis (the bracelets basis) that is positive, in the sense that the structure constants for multiplication are positive integers.

character variety | canonical basis | Jones polynomial | cluster algebras

1. Introduction

For a compact oriented surface Σ (possibly with boundary), the *Kauffman bracket skein algebra*, denoted $\mathrm{Sk}_q(\Sigma)$, is the $\mathbb{Z}[q^{\pm 1}]$-module spanned by framed links in $\Sigma \times [0,1]$ modulo the local relations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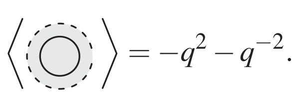
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Vertical stacking of links makes $\mathrm{Sk}_q(\Sigma)$ into an algebra: to form $\langle D_1 \rangle \cdot \langle D_2 \rangle$, superimpose D_1 onto D_2, making D_1 cross over D_2.

This skein algebra was first defined by Przytycki (1) and Turaev (2) as an extension of the Jones polynomial of knots in S^3 to knots in a surface cross an interval. When specialized to $q = \pm 1$, we no longer need to record crossing information. For $q = -1$, we essentially get the algebra of functions on the $\mathrm{SL}_2(\mathbb{R})$ character variety of Σ (3–5). A choice of a spin structure gives an isomorphism between $\mathrm{Sk}_q(\Sigma)$ and $\mathrm{Sk}_{-q}(\Sigma)$ (6). More naturally, $\mathrm{Sk}_1(\Sigma)$ can be thought of as the algebra of functions on the twisted $\mathrm{SL}_2(\mathbb{R})$ character variety.

Definition 1.1. *A* twisted $\mathrm{SL}_2(\mathbb{R})$ representation *of a surface* Σ *is a representation of* $\pi_1(UT\Sigma)$*, the fundamental group of the unit tangent bundle of* Σ*, into* $\mathrm{SL}_2(\mathbb{R})$*, with the property that rotation by* 2π *acts by* $-1 \in \mathrm{SL}_2(\mathbb{R})$*. The* twisted $\mathrm{SL}_2(\mathbb{R})$ character variety *is the algebrogeometric quotient of twisted* $\mathrm{SL}_2(\mathbb{R})$ *representations by conjugation.*

A hyperbolic structure on Σ gives a canonical twisted $\mathrm{SL}_2(\mathbb{R})$ representation. See, e.g., proposition 10 in ref. 7.

In this paper, we are mainly interested in $\mathrm{Sk}_1(\Sigma)$, henceforth denoted $\mathrm{Sk}(\Sigma)$. Our main result is that it has a *positive basis*.

Definition 1.2. *For an algebra A over* $\mathbb{Z}$ *(free as a* $\mathbb{Z}$*-module), a basis* $\{x_i\}$ *is* positive *if*

$$x_i \cdot x_j = \sum_k m_{ij}^k x_k,$$

where $m_{ij}^k \geq 0$.

We will show that the *bracelets basis* (*Definition 4.9*) of the skein algebra is positive. This basis is *not* made of crossingless diagrams. In Fig. 1, instead of *bangles* we use *bracelets*.

Theorem 1. *The bracelets basis is a natural positive basis for* $\mathrm{Sk}(\Sigma)$.

The basis is *natural* in the sense that it is invariant under the mapping class group (automorphisms of the surface). Although a spin structure gives an isomorphism between $\mathrm{Sk}_1(\Sigma)$ and $\mathrm{Sk}_{-1}(\Sigma)$ as algebras, it is unlikely that $\mathrm{Sk}_{-1}(\Sigma)$ has a natural positive basis, as Σ generally does not have a canonical spin structure.

We work with a mild extension of the skein algebra, to include marked points and arcs with endpoints on the marked points. This algebra includes many of the cluster variables in the cluster algebra associated with a surface (8–10), namely the cluster variables without notched arcs. In particular, for a surface without punctures, this gives a positive, natural basis for an algebra between the cluster algebra and the upper cluster algebra of the surface.

Positivity was first conjectured by Fock and Goncharov in their ground-breaking paper (section 12 in ref. 11). The bracelets basis was considered by Musiker, Schiffler, and Williams (12), who proved a weaker form of positivity. This weaker positivity and explicit combinatorial formulas have been well studied (13–16).

In fact, we will prove a stronger theorem.

Theorem 2. *For any diagram D on* Σ*, the expansion of* $\langle D \rangle$ *in the bracelets basis is sign-coherent. If D has no null-homotopic components or nugatory crossings, then the expansion is positive.*

Here, *sign-coherent* means that either all terms are positive or all are negative. A *nugatory crossing* is a crossing that cuts off a null-homotopic loop.

The proof proceeds by carefully picking a crossing to resolve by Eq. **1**, being careful to avoid ever introducing a negative sign.

Extensions and Future Work. *Theorem 1* suggests many possible extensions. First, can anything be said when $q \neq 1$?

Conjecture 1.3. *The bracelets basis can be lifted to a positive basis for the quantum skein algebra* $\mathrm{Sk}_q(\Sigma)$.

See *Conjecture 4.20* for a more precise version. This conjecture was essentially made by Fock and Goncharov (conjecture 12.4 in ref. 11). The techniques in this paper will not work for the q-deformation, as there is no obvious analog of *Theorem 2* for the quantum skein algebra.

Fig. 1 suggests another natural basis for $\mathrm{Sk}_q(\Sigma)$, the *bands basis*.

Question 1.4. *When is the bands basis positive?*

See *Conjecture 4.19* for a possible answer.

Finally, the existence of a positive basis suggests the presence of a "nice" categorification, where product becomes a monoidal tensor product and sum becomes direct sum, or possibly a composition series. We leave the precise formulation vague.

Significance

The Jones polynomial of knots is one of the simplest and most powerful knot invariants, at the center of many recent advances in topology; it is a polynomial in a parameter *q*. The skein algebra of a surface is a natural generalization of the Jones polynomial to knots that live in a thickened surface. In this paper, we propose a basis for the skein algebra. This basis has positivity properties when *q* is set to 1, and conjecturally for general values of *q* as well. This is part of a more general conjecture for cluster algebras, and suggests the existence of well-behaved higher-dimensional structures.

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MATHEMATICS

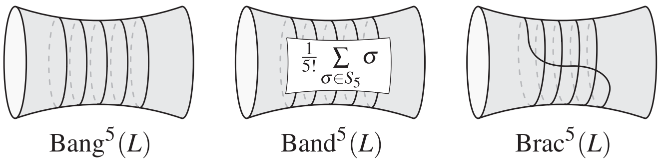


Fig. 1. Examples of bangle, band, and bracelet operations applied to the core loop of an annulus. The bangle has parallel copies, the band averages over all ways of joining, and the bracelet wraps multiple times.

Question 1.5. *Is there a monoidal categorification of* $\mathrm{Sk}_1(\Sigma)$ *or* $\mathrm{Sk}_q(\Sigma)$ *that makes positivity of the bracelets basis or bands basis manifest?*

2. Preliminaries on Surfaces and Curves

We first collect some basic facts on the topology of curves in surfaces.

Definition 2.1. *A* marked surface $\Sigma = (\mathbf{S}, \mathbf{M})$ *is a pair of a surface* $\mathbf{S}$, *possibly with boundary, and a finite set* $\mathbf{M} \subset \mathbf{S}$ *of marked points. Marked points in the interior of* $\mathbf{S}$ *are called* punctures. *When we think of* Σ *itself as a topological space, we mean* $\mathbf{S}\backslash\mathbf{M}$. *For convenience, we will assume that* $\mathbf{S}$ *is connected*.

Simple surfaces with few or no marked points are not excluded.

Definition 2.2. *A* (curve) diagram $D = (X, \phi)$ *on* Σ *is 1-manifold with boundary,* X, *and an immersion* $\phi : (X, \partial X) \to (\mathbf{S}, \mathbf{M})$, *by which we mean an immersion of* X *in* $\mathbf{S}$ *so that each boundary point of* X *maps to a marked point and no point in the interior of* X *maps to a marked point. We also require that* D *has only simple transverse crossings, and that no point in the interior of* X *maps to* $\partial\mathbf{S}$. D *is* connected *if* X *is connected,* D *is an arc if* X *is an interval, and* D *is a* loop *if* X *is a circle.*

There are several equivalence relations on diagrams.

Definition 2.3. *A* Reidemeister move *is one of the moves in Fig. 2 (in either direction). Note that this includes some moves that change the number of components. A Reidemeister* reduction *is a Reidemeister move that reduces or keeps constant the number of intersections and components, i.e., all moves from left to right, and RIII in either direction. A* strict reduction *is any reduction other than RIII*.

Definition 2.4. *Two diagrams* D_1, D_2 *are* ambient isotopic *if they can be related by an isotopy of* Σ, *or equivalently if there is an isotopy of* D_1 *to* D_2 *that does not change any of the crossings. We always consider diagrams up to ambient isotopy; the set of diagrams up to ambient isotopy is denoted* $\mathbf{D}(\Sigma)$.

D_1 *and* D_2 *are* regular isotopic *if they can be connected by a path within the space of immersions, dropping the condition on crossings but keeping the interior of the diagram disjoint from* $\partial\mathbf{S} \cup \mathbf{M}$. *Equivalently,* D_1 *can be connected to* D_2 *using moves RII, RIIb, and RIII. The set* $\mathbf{D}(\Sigma)$ *modulo regular isotopy is denoted* $\mathbf{C}(\Sigma)$.

D_1 *and* D_2 *are* homotopic, *written* $D_1 \sim D_2$, *if they can be connected within the space of all continuous maps. Equivalently,* D_1 *can be connected to* D_2 *using moves RI, RII, RIIb, and RIII.*

D_1 reduces *to* D_2 *if* D_1 *can be turned in to* D_2 *by a chain of zero or more Reidemeister reductions, of any of the types.*

There is a product on diagrams modulo regular isotopy.

Definition 2.5. *For* $C_1, C_2 \in \mathbf{C}(\Sigma)$, *their* product C_1C_2 *is obtained by taking diagrams* D_1, D_2 *so that* $C_i = [D_i]$ *and* D_1 *intersects* D_2 *transversally, and defining*

$$C_1C_2 = [D_1 \cup D_2].$$

Lemma 2.6. *The product above is well defined, and makes* $\mathbf{C}(\Sigma)$ *into a commutative, associative monoid with unit the empty diagram.*

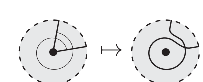
Proof: Standard. □

We next give conditions for a diagram to have minimal self-intersections.

Definition 2.7. *For a diagram* $D = (X, \phi)$, *a* segment *of* D *is an oriented subinterval of* X *whose endpoints are either endpoints of* X *or preimages of crossings. If two segments* S_1 *and* S_2 *meet at a marked point p, then a* turn *from* S_1 *to* S_2 *at p is a homotopy class of arcs from* $\phi(S_1)$ *to* $\phi(S_2)$ *inside* $N\backslash p$, *where* N *is a small neighborhood of p. (If p is not a puncture, there is only one such turn.)*

A k-chain *of D is a sequence of k segments* $(S_i)_{i=1}^k$ *so that the end of* S_i *has the same image as the start of* S_{i+1}, *with i interpreted modulo k, and so that the* S_i *are disjoint subsets of X. Furthermore choose a turn from* S_i *to* S_{i+1} *whenever they meet at a marked point. As a special case, a* 0-chain *of D is a loop component of D.*

For any k-chain H, there is an associated loop H°, *obtained by smoothing out the corners at the endpoints of the segments. At marked points, follow the chosen turn without nugatory crossings:*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An embedded k-gon *is a k-chain H so that* H° *bounds a disk. A* singular k-gon *or just* k-gon *is a k-chain H so that* H° *is null-homotopic. Finally, a* weak segment *is an immersed subinterval S of X whose endpoints are crossings of D or endpoints of X. (That is, S may wrap more than once around a loop component of D.) A* weak k-chain *is like a k-chain, but using weak segments and dropping the requirement that the* S_i *be disjoint, and a* weak k-gon *is a weak k-chain H so that* H° *is null-homotopic. (For H a weak k-chain,* H° *is defined analogously to the above definition, possibly running multiple times over some portions of D.) We also use the terms disks, monogons, and bigons for 0-gons, 1-gons, and 2-gons (Fig. 3)*.

To relate chains to the fundamental group, note that a representative of $\alpha \in \pi_1(\Sigma, x)$ *can be viewed as a 1-chain, so* α° *is a loop on* Σ. *Conversely, the* holonomy *of a loop L on* Σ *is the corresponding element of* $\pi_1(\Sigma, x)$, *where we connect L to the basepoint x by a specified path. [The term "holonomy" comes from thinking about the canonical* $\pi_1(\Sigma)$ *bundle over* Σ*.]*

For example, the left-hand side of

- RO has an embedded disk,
- RI and RIb have an embedded monogon,
- RII and RIIb have an embedded bigon, and
- RIII has an embedded triangle.

Definition 2.8. *In a diagram D, a* trivial component *is a null-homotopic component, necessarily either a loop or an arc from a marked point to itself. D is* simple *if it has no crossings and no trivial components. D is* taut *if it has a minimum number*

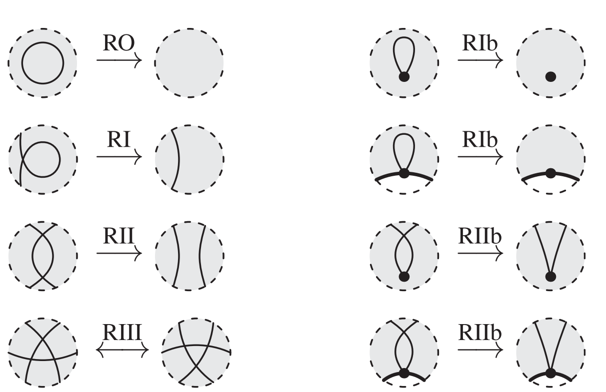


Fig. 2. The Reidemeister moves. The diagrams show a local portion of the surface. The surface is shaded gray; white regions are not in the surface. In moves RIb and RIIb, there may be more arcs ending at the marked point, not intersecting the displayed arcs.

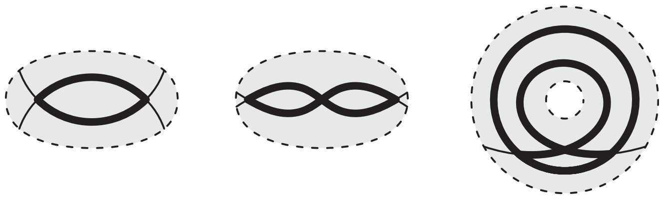


Fig. 3. Embedded, singular, and weak bigons. Two portions of the weak bigon run over the same part of the loop.

of self-intersections in its homotopy class and has no trivial components.

Lemma 2.9. *If $D = D_1 \cup D_2$ is a taut diagram, then D_1 and D_2 are also taut diagrams.*

Proof: This follows, for instance, from the fact that a diagram is taut if and only if it is length-minimizing with respect to some metric (17, 18). □

Diagrams can be monotonically simplified.

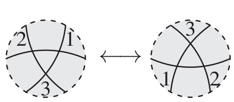
Theorem 2.10. (Hass and Scott, ref. 19). *An arbitrary diagram can be turned into a taut diagram by a sequence of Reidemeister reductions.*

(Hass and Scott proved a version of this theorem without arcs or marked points, but the techniques extend immediately.)

Definition 2.11. *For D a diagram, the set of crossings of D is denoted* Cross(D). *For $x \in$* Cross(D), *a* resolution *of x is one of the two diagrams obtained by replacing a neighborhood of x by a local picture without crossings. A* connected *resolution is one that does not increase the number of components of D.*

Lemma 2.12. *Let S be a set of Reidemeister reductions containing RII. If D_1 reduces to D_2 by a set of reductions from S, then there is an injection $m : \mathrm{Cross}(D_2) \to \mathrm{Cross}(D_1)$ so that for each $x \in \mathrm{Cross}(D_2)$, the resolutions of $m(x)$ are related by moves in* S *to resolutions of x.*

Proof: It suffices to consider the case when D_1 and D_2 are related by a single Reidemeister reduction. When $x \in \mathrm{Cross}(D_2)$ is outside of the reducing region, $m(x)$ is the same crossing as an element of Cross(D_1). The only case where x can be inside the reducing region is for RIII, where m is the correspondence.



In this RIII case, the resolution at x and the resolution at $m(x)$ are related by two RII moves. Otherwise, the resolutions are related by the same Reidemeister move that relates D_1 and D_2. □

Lemma 2.13. *A taut diagram has no singular disks or monogons, and the only singular bigons are bigons between isotopic arcs.*

Proof: Let D be a taut diagram. It follows by definition that D has no singular disk. If D has a singular monogon, then the connected resolution of the crossing of the monogon yields a diagram D' homotopic to D but with fewer crossings. Similarly, if D has a singular bigon involving at least one crossing, resolving the crossing(s) yields a simpler diagram $D' \sim D$. □

There is also a partial converse to *Lemma 2.13*.

Theorem 2.14. *If a diagram D is not taut, then it has a singular disk, singular monogon, or weak bigon.*

Proof: This was first proved by Hass and Scott in 1985 (theorem 3.5 in ref. 20). We give a short proof using their later curve-shortening techniques (19), as we will use the technique for *Lemma 2.21* below. By *Theorem 2.10*, there is a sequence of Reidemeister reductions

$$D = D_0 \to D_1 \to \cdots \to D_n,$$

where D_n is taut. Since D_0 is not taut, at least one of these reductions, say $D_k \to D_{k+1}$, is strictly reducing. If $D_k \to D_{k+1}$ is

RO, D_k has an embedded disk;

RI, D_k has an embedded monogon; or

RII or RIIb, D_k has an embedded bigon.

In each case, we follow the disk, monogon, or bigon backward from D_k to D_0, using *Lemma 2.15*. □

Lemma 2.15. *If a diagram D_1 reduces to D_2 and D_2 has a singular disk, singular monogon, or weak bigon, then D_1 has one as well.*

Proof: We use the map $m : \mathrm{Cross}(D_2) \to \mathrm{Cross}(D_1)$ from *Lemma 2.12*.

- If D_2 has a singular disk, the homotopy type of this component is the same in D_1.
- If D_2 has a singular monogon with corner x, then $m(x)$ is the corner of a singular monogon for D_1.
- If D_2 has a weak bigon with corners at x and y, then D_1 has a weak bigon with corners at $m(x)$ and $m(y)$. □

Remark 2.16. Theorem 2.14 *is false if we replace "weak bigon" with "singular bigon" in the statement. Let L be a simple loop and* $\mathrm{Brac}^3(L)$ *the loop that wraps 3 times around L. Then there is a nontaut embedding of* $L \cup \mathrm{Brac}^3(L)$ *with a weak bigon but no singular bigon (20). Fig. 3 on the right shows an example of problematic local behavior just after a singular bigon has become a weak bigon. On the other hand, a taut diagram may have a weak bigon; for instance,* $\mathrm{Brac}^2(L)$ *always has a weak bigon. So* Theorem 2.14 *does not give necessary and sufficient conditions for a diagram to be taut.*

Despite *Remark 2.16*, there is a converse for connected diagrams.

Theorem 2.17. (lemma 4.1 and theorem 4.2 in ref. 20). *If a connected diagram is not taut, then it has a singular disk, monogon, or bigon.*

Remark 2.18. *It is possible to give a curve-shortening proof of* Theorem 2.17 *along the lines of the proof of* Theorem 2.14 *above. The bad behavior in Fig. 3 cannot occur when the diagram is connected.*

Bracelets, loops that wrap multiple times, play a key role in our construction.

Proposition 2.19. *If A and B are two taut diagrams with holonomy α and β, B is simple, and $\alpha = \beta^k$, then A has $k-1$ self-intersections. We can label these self-intersections x_i, $1 \le i < k$, so the holonomy of the two components of the disconnected resolution of A at x_i are conjugate to β^i and β^{k-i}.*

Proof: Lemma 1.9 in ref. 20 and theorem 2.1 (case 3) in ref. 19. □

Theorem 2 says that taut diagrams have positive expansions in the bracelets basis. There is a more general class of diagrams with positive expansions.

Definition 2.20. *A diagram is* weakly positive *if it is regular isotopic to a taut diagram. A diagram is* strongly positive *if it has no singular monogons or singular disks.*

Lemma 2.21. *A strongly positive diagram is weakly positive.*

Proof: Let D be strongly positive. By *Theorem 2.10*, D can be reduced to a taut diagram D' using reductions. By *Lemma 2.15*, if at any step along the way we use an RO move, then the singular disk can be followed back to give a singular disk for D; but D is strongly positive, so this cannot happen. Similarly for singular monogons and RI or RIb moves. Thus, we only use moves RII, RIIb, and RIII in the reduction from D to D', as desired. □

Finally, roots are unique in $\pi_1(\Sigma)$.

Lemma 2.22. *For γ, δ nontrivial elements of $\pi_1(\Sigma)$ and $k, l \in \mathbb{Z}_{>0}$, if $\gamma^k = \delta^l$ then there is an element η so that $\gamma = \eta^{\mathrm{lcm}(k,l)/k}$ and $\delta = \eta^{\mathrm{lcm}(k,l)/l}$.*

One can prove this, for instance, by taking a hyperbolic or flat metric on Σ and taking the geodesic representative for the conjugacy class of $\gamma^k = \delta^l$, which will multiply cover a well-defined primitive loop L. Then γ and δ must both be conjugate to powers of the monodromy of L.

3. Skein Algebra

The skein algebra $\mathrm{Sk}(\Sigma)$ is the quotient of $\mathbb{Z}\mathbf{D}(\Sigma)$ by the relations from Fig. 4. More precisely, we make the following definitions.

Definition 3.1. *For $D \in \mathbf{D}(\Sigma)$, a* skein reduction *of D is the element of $\mathbb{Z}\mathbf{D}(\Sigma)$ obtained by one of the replacements illustrated in Fig. 4:*

Crossing resolution (C): *Replace a crossing by the sum of its two resolutions.*

Unknot removal (U): *Replace an embedded* 0*-gon by $-2\times$ the rest of D.*

Punctured disk removal (P): *Replace a simple loop surrounding a puncture by $2\times$ the rest of D.*

Monogon removal (M): *Replace an embedded monogon at a marked point by* 0.

In each case, the reduction disk *is the disk indicated in Fig. 4. For reductions* (*C*), (*U*), *and* (*P*) *the intersection of D with the reduction disk is as shown; for* (*M*) *there may be other arcs ending at the marked point.*

Similarly, if D skein reduces to z and $z_3 \in \mathbb{Z}\mathbf{D}(\Sigma)$ does not have a term involving D, we also say that $\langle D\rangle + z_3$ has an elementary skein reduction to $z + z_3$. Say that z_1 skein reduces *to z_2 if they differ by a sequence of zero or more elementary skein reductions, always going from left to right.*

Definition 3.2. *The* skein algebra $\mathrm{Sk}(\Sigma)$ *is the quotient of $\mathbb{Z}\mathbf{D}(\Sigma)$ by skein reductions. Let $\langle D\rangle$ be the image of a diagram D in* $\mathrm{Sk}(\Sigma)$.

Proposition 3.3. *If D_1 and D_2 are regular isotopic, then $\langle D_1\rangle = \langle D_2\rangle$. If D_1 differs from D_2 by an RI move, then $\langle D_1\rangle = -\langle D_2\rangle$.*

Proof: Expand both sides using skein reductions. □

Proposition 3.3 lets us talk about the skein class $\langle C\rangle$ of a curve C.

Proposition 3.4. *On a punctured surface Σ, union of curves induces a commutative, associative product on* $\mathrm{Sk}(\Sigma)$.

Proof: As noted in *Lemma 2.6*, there is a product on $\mathbf{C}(\Sigma)$. By *Proposition 3.3*, this gives a map $\mathbb{Z}\mathbf{C}(\Sigma) \times \mathbb{Z}\mathbf{C}(\Sigma) \to \mathrm{Sk}(\Sigma)$. We must show that for $C_1, C_1' \in \mathbb{Z}\mathbf{C}(\Sigma)$, if $\langle C_1\rangle = \langle C_1'\rangle$, then $\langle C_1 C_2\rangle = \langle C_1' C_2\rangle$. It suffices to consider the case that C_1 is a single diagram (mod regular isotopy) and C_1 and C_1' differ by an elementary reduction. For reductions (C), (U), and (M), we may assume by ambient isotopy of C_2 that the reduction disk in C_1 does not intersect C_2, so the reduction descends. For reduction (P), we cannot avoid arcs of C_2 that end at the enclosed puncture, but a direct computation shows that the two sides have the same value. □

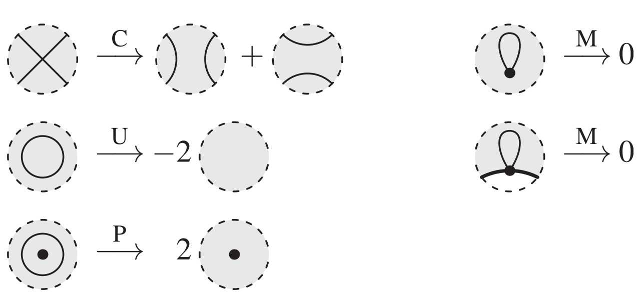


Fig. 4. The relations in $\mathrm{Sk}(\Sigma)$, when Σ is a surface with punctures. The relation in each case is that the left-hand side equals the right-hand side, but we draw an arrow to indicate that it is a reduction as we go from left to right. For reduction (M) there can be any number of other arcs ending at the pictured marked point.

Definition 3.5. *The* complexity *of a diagram D is the pair*

$$(c(D), r(D)),$$

where $c(D)$ is the number of crossings of D and $r(D)$ is the number of reducible components of D. A reducible component *of D is a place where moves* (*U*), (*P*), *or* (*M*) *can be applied.*

These pairs are ordered with the lexicographic order. The complexity of a linear combination of diagrams is the list of complexities of the nonzero terms, sorted in decreasing order, and considered with the lexicographic order on lists.

A diagram is irreducible *if its complexity is* $(0, 0)$. *Irreducible diagrams are nearly the same as simple diagrams, except that loops around punctures are forbidden.*

Lemma 3.6. *Complexity strictly reduces under skein reductions.*

Proof: This is immediate for reductions of a single diagram, and follows easily for linear combinations. □

Lemma 3.7. *Any sequence of skein reductions in $\mathbb{Z}\mathbf{D}(\Sigma)$ eventually terminates.*

Proof: The ordering on complexities is a well ordering. □

Proposition 3.8. *Irreducible diagrams form a basis for* $\mathrm{Sk}(\Sigma)$. *Any element of* $\mathrm{Sk}(\Sigma)$ *can be expressed in this basis by applying skein reductions whenever possible, in any order.*

Proof: *Lemma 3.7* implies that irreducible curves span $\mathrm{Sk}(\Sigma)$. To see linear independence, observe that reductions satisfy a *diamond property*: if $z_1, z_2, z_3 \in \mathbb{Z}\mathbf{D}(\Sigma)$ are such that z_1 has an elementary skein reduction to both z_2 and z_3, then there is another element $z_4 \in \mathbb{Z}\mathbf{D}(\Sigma)$ so that both z_2 and z_3 skein reduce to z_4. (This is very easy for this skein algebra, since the reducing disks for different relations can almost never overlap, except for monogons at the same puncture.) Then the diamond lemma (21) implies that any two sequences of skein reductions terminate at the same place. □

Remark 3.9. *The reduction* (*P*) *may look unfamiliar to readers used to the Jones skein. This value for the loop is forced by consistency with multiplication* (Proposition 3.4). *The quantum analog is not obvious, and requires introducing* opened surfaces *(cf. ref. 22).*

4. Three Bases

Proposition 3.8 gives a basis for the skein algebra, but this basis is not always positive in the sense of *Definition 1.2*. There are, in fact, three related bases. We first give the elementary building blocks.

Definition 4.1. *For a simple loop $L \in \mathbf{D}(\Sigma)$ with holonomy $\gamma \in \pi_1(\Sigma)$ and an integer $k > 0$, we define three ways to create an element of $\mathbb{Q}\mathbf{D}(\Sigma)$, as in Fig. 1.*

- $\mathrm{Bang}^k(L)$, *the k'th* bangle *of L, is L^k, i.e., k parallel copies of L.*
- *Let $I \subset L$ be a short interval. Then* $\mathrm{Band}^k(L)$, *the k'th* band *of L, is k copies of $L \backslash I$ with the ends connected by averaging all $k!$ ways of pairing the endpoints on the two sides.*
- $\mathrm{Brac}^k(L)$, *the k'th* bracelet *of L, is the loop whose holonomy is γ^k, embedded tautly. By* Proposition 2.19, $\mathrm{Brac}^k(L)$ *has $k-1$ self-intersections.*

For convenience, for a simple loop L and simple arc A, also define

$$\begin{aligned}\mathrm{Bang}^0(L) &= 1\\ \mathrm{Bang}^k(A) &= \langle A\rangle^k\\ \mathrm{Band}^0(L) &= 1\\ \mathrm{Band}^k(A) &= \langle A\rangle^k\\ \mathrm{Brac}^0(L) &= 2\\ \mathrm{Brac}^k(A) &= \langle A\rangle^k.\end{aligned}$$

A priori, $\mathrm{Band}^k(L)$ is only in $\mathrm{Sk}_{\mathbb{Q}}(\Sigma) := \mathrm{Sk}(\Sigma) \otimes \mathbb{Q}$. In fact, it is in $\mathrm{Sk}(\Sigma)$, i.e., the coefficients are integral after reducing modulo the skein relations (*Proposition 4.8*).

Example 4.2. *If* $\langle L\rangle = z$, *then*

$$\begin{aligned}\mathrm{Bang}^2(L) &= z^2\\ \mathrm{Band}^2(L) &= z^2-1\\ \mathrm{Brac}^2(L) &= z^2-2.\end{aligned}$$

The first equation is trivial. The third equation follows by applying the skein relation at the unique crossing of $\mathrm{Brac}^2(L)$. *The second equation is the average of the other two.*

Definition 4.3. *The* Chebyshev polynomials of the first kind *are polynomials* $T_n(z)$ *satisfying the recurrence*

$$T_0(z) = 2 \quad [4]$$

$$T_1(z) = z \quad [5]$$

$$T_{n+1}(z) = zT_n(z) - T_{n-1}(z). \quad [6]$$

They satisfy

$$T_k(z)T_l(z) = T_{k+l}(z) + T_{|k-l|}(z) \quad [7]$$

$$T_k(e^x + e^{-x}) = e^{kx} + e^{-kx}. \quad [8]$$

The Chebyshev polynomials of the second kind *are polynomials* $U_n(z)$ *satisfying the recurrence*

$$U_0(z) = 1 \quad [9]$$

$$U_1(z) = z \quad [10]$$

$$U_{n+1}(z) = zU_n(z) - U_{n-1}(z). \quad [11]$$

They satisfy

$$U_k(z)U_l(z) = U_{k+l}(z) + U_{k+l-2}(z) + \ldots + U_{|k-l|}(z) \quad [12]$$

$$U_k(e^x + e^{-x}) = e^{kx} + e^{(k-2)x} + \ldots + e^{-kx}. \quad [13]$$

Proposition 4.4. *For any simple loop L and integer* $n > 0$,

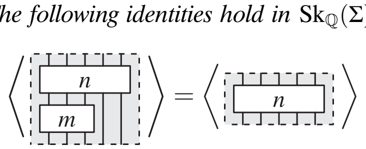
$$\mathrm{Brac}^n(L) = T_n(\langle L\rangle). \quad [14]$$

Proof: This is trivially true for $n = 1$ and the $n = 2$ case was done in *Example 4.2*. To compute $\mathrm{Brac}^n(L)$ with $n > 2$, resolve one of the two outer crossings of $\mathrm{Brac}^n(L)$ (with holonomies γ and γ^{n-1}, as in *Proposition 2.19*). One of the two resolutions is $L \cdot \mathrm{Brac}^{n-1}(L)$. The other resolution differs by an RI move from $\mathrm{Brac}^{n-2}(L)$. This gives Eq. **6** in $\mathrm{Sk}(\Sigma)$. □

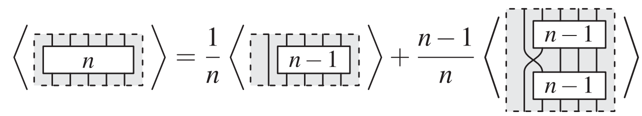
Remark 4.5. *By resolving* $\mathrm{Brac}^n(L)$ *at other crossings, we get a short proof of Eq. 7 as applied to bracelets.*

To find the analog of *Proposition 4.4* for the bands basis, we introduce the graphical notation that a box with an n inside means averaging over all ways of joining the n strands on the two sides of the box. Diagrams should be interpreted as having a variable number of strands (including 0), as indicated in the boxes.

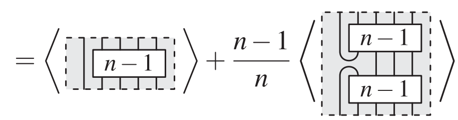
Lemma 4.6. *The following identities hold in* $\mathrm{Sk}_{\mathbb{Q}}(\Sigma)$*:*



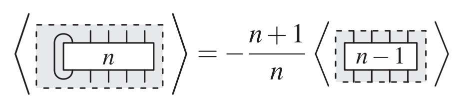
[15]



[16]



[17]



[18]

Proof: Eq. **15** is true because averaging twice is the same as averaging once. To see Eq. **16**, note that if we average over S_n, the first strand on the top is connected to the first strand on the bottom with probability $1/n$, and connected somewhere else with probability $(n-1)/n$. These two possibilities correspond to the two terms on the right. Applying the skein relation at the crossing gives Eq. **17**. Eq. **18** follows from Eq. **17** by taking a partial trace: join the first strand on the top to the first strand on the bottom. □

Remark 4.7. Lemma 4.6 *is the* $q = 1$ *specialization of standard equations for the Jones–Wenzl idempotents (23)*.

Proposition 4.8. *For any simple loop L and integer* $n > 0$,

$$\mathrm{Band}^n(L) = U_n(\langle L\rangle). \quad [19]$$

Proof: This is trivial for $n = 1$ and already checked for $n = 2$. For $n > 2$, by Eqs. **17**, **15**, and **18** (in that order) to the diagram defining $\mathrm{Band}^n(L)$, we find that $\mathrm{Band}^n(L)$ satisfies Eq. **11**. □

Definition 4.9. *The* bangles basis $\mathbf{B}_0(\Sigma)$ *for* $\mathrm{Sk}(\Sigma)$ *consists of all irreducible diagrams. It is parameterized by* integer laminations, *which are unordered collections* $\mu = \{(a_i, C_i)\}_{i=1}^k$, *where*

- *each* a_i *is a positive integer,*
- *each* C_i *is a connected irreducible diagram,*
- *no two* C_i *intersect, and*
- *all of the* C_i *are distinct up to ambient isotopy.*

The corresponding basis element in $\mathbf{B}_0(\Sigma)$ *is*

$$\mathrm{Bang}(\mu) := \prod_{i=1}^{k} \mathrm{Bang}^{a_i}(C_i) = \prod_{i=1}^{k} \langle C_i\rangle^{a_i}.$$

The bands *and* bracelets bases $\mathbf{B}_1(\Sigma)$ *and* $\mathbf{B}_2(\Sigma)$ *are also parameterized by integer laminations, but with corresponding basis elements*

$$\mathrm{Band}(\mu) := \prod_{i=1}^{k} \mathrm{Band}^{a_i}(C_i) \qquad \mathrm{Brac}(\mu) := \prod_{i=1}^{k} \mathrm{Brac}^{a_i}(C_i),$$

respectively. Informally, start from an irreducible diagram and replace parallel loops with the corresponding band or bracelet.

Proposition 4.10. *The sets* $\mathbf{B}_0(\Sigma)$, $\mathbf{B}_1(\Sigma)$, *and* $\mathbf{B}_2(\Sigma)$ *each form a basis for* $\mathrm{Sk}(\Sigma)$.

Proof: For $\mathbf{B}_0(\Sigma)$ this is Proposition 3.8. The other two bases are triangular with respect to $\mathbf{B}_0(\Sigma)$. □

Here is an intrinsic characterization of the bracelets basis.

Definition 4.11. *A* bracelet loop *is a loop of the form* $\mathrm{Brac}^k(L)$ *for some* $k \geq 1$ *and some simple loop* L. *A* multibracelet *is a diagram* D *in which*

- *each component is a simple arc or a bracelet loop, and*
- *no two components intersect.*

A multibracelet has parallel bracelets *if there are two components that are bracelets of the same simple loop.*

Lemma 4.12. *A diagram is in* $\mathbf{B}_2(\Sigma)$ *if and only if it is a multibracelet with no parallel bracelets and no bracelets of punctured disks.*

Examples on Positivity. *Theorem 1* says that the bracelets basis is positive. We will now give some other examples and conjectures on positivity and nonpositivity.

Example 4.13. *Let* Σ *be the annulus with two marked points, one on each boundary component. Let* A_k *be the arc connecting the two marked points at slope* k *(i.e., wrapping* $|k|$ *times around the core loop, clockwise or counterclockwise according to the sign of* k*). Let* L *be the core loop, and let* B *be a push-off of the union of the two boundary components (considered as a diagram with two arcs). Then elementary induction using the skein rules shows that*

$$\langle \mathrm{Brac}^n(L)\rangle \cdot \langle A_k\rangle = \langle A_{k+n}\rangle + \langle A_{k-n}\rangle \qquad [20]$$

$$\langle A_a\rangle \cdot \langle A_{a+k}\rangle = \langle B\rangle \sum_{i=1}^{\lfloor k/2\rfloor} i\,\mathrm{Brac}^{k-2i}(L) + \begin{cases} \left\langle A_{a+\frac{k}{2}}\right\rangle^2 & k \text{ even} \\ \left\langle A_{a+\frac{k-1}{2}}\right\rangle\left\langle A_{a+\frac{k+1}{2}}\right\rangle & k \text{ odd.}\end{cases} \qquad [21]$$

The structure constants are positive, as expected. They are also positive in the bands basis, but in the bangles basis,

$$\langle A_0\rangle\langle A_5\rangle = \langle A_2\rangle\langle A_3\rangle + \langle B\rangle\left(\langle L\rangle^3 - \langle L\rangle\right).$$

Example 4.14. *Take* $\Sigma = T^2$, *the unpunctured torus. Let* α, β *be the two generators for* $\pi_1(T^2)$, *and for* $\gamma \in \pi_1(T^2)$, $\gamma \neq 1$, *let* $\gamma^\circ \in \mathbf{D}(\Sigma)$ *be a taut representative for the corresponding conjugacy class. Set* $C_{a,b} = (\alpha^a\beta^b)^\circ$ *for* $a, b \in \mathbb{Z}$, *not both* 0. *The bracelets basis is*

$$\mathbf{B}_2(T^2) = \left\{\langle C_{a,b}\rangle \middle| (a,b) \neq (0,0)\right\} \cup \{1\}.$$

The only duplicates on the list above arise from the equality $\langle C_{a,b}\rangle = \langle C_{-a,-b}\rangle$. *For convenience, also define* $\langle C_{0,0}\rangle = 2$.

Proposition 4.15. *The multiplication rules for* $\mathrm{Sk}(T^2)$ *are*

$$\langle C_{a,b}\rangle \cdot \langle C_{c,d}\rangle = \langle C_{a+c,b+d}\rangle + \langle C_{a-c,b-d}\rangle. \qquad [22]$$

Proof: This follows from ref. 24, but we give a short independent argument. If $(a,b) = \pm(c,d)$, Eq. **23** follows from *Proposition 4.4*. Otherwise, take a taut embedding of $(\alpha^a\beta^b)^\circ \cup (\alpha^c\beta^d)^\circ$ and resolve any one crossing between the two components. Both resolutions are strongly positive (see *Lemma 5.10* below) and are $(\alpha^{a+c}\beta^{b+d})^\circ$ and $(\alpha^{a-c}\beta^{b-d})^\circ$. □

Corollary 4.16. *The bands basis and the bangles basis are not positive on* T^2.

Proof: *Proposition 4.15* tells us that

$$\begin{aligned}\langle C_{0,1}\rangle\langle C_{2,1}\rangle &= \mathrm{Brac}^2(C_{1,0}) + \mathrm{Brac}^2(C_{1,1}) \\ &= \mathrm{Band}^2(C_{1,0}) + \mathrm{Band}^2(C_{1,1}) - 2 \\ &= \mathrm{Bang}^2(C_{1,0}) + \mathrm{Bang}^2(C_{1,1}) - 4.\end{aligned}$$

□

Example 4.17. *We can do some computations for* $\Sigma = T^2 \backslash D^2$, *a perforated torus. Let* B *be the loop around the boundary. For* $(k,l) \in \mathbb{Z}^2$ *relatively prime, let* $C_{k,l}$ *be the simple loop at slope* k/l *(i.e., homotopic to* $\alpha^k\beta^l$*) on the unpunctured torus. (There is a unique such loop.) The first interesting product is*

$$\begin{aligned}\langle C_{0,1}\rangle\langle C_{2,1}\rangle &= \langle C_{1,1}\rangle^2 + \langle C_{0,1}\rangle^2 + \langle B\rangle - 2 \\ &= \mathrm{Band}^2(C_{1,1}) + \mathrm{Band}^2(C_{0,1}) + \langle B\rangle \\ &= \mathrm{Brac}^2(C_{1,1}) + \mathrm{Brac}^2(C_{0,1}) + \langle B\rangle + 2.\end{aligned}$$

This is positive in the bracelets and bands bases, but not for the bangles basis, and reduces to the answer from Eq. **22** *if we set* $\langle B\rangle = -2$.

Problem 4.18. *Find a general formula for* $\langle C_{a,b}\rangle\langle C_{c,d}\rangle$ *on* $T^2\backslash D^2$.

The examples above imply that the bangles basis is almost never positive. On the other hand, the following conjecture is plausible.

Conjecture 4.19. *The bands basis is positive when* Σ *has either nonempty boundary or at least one puncture, i.e., if* $\pi_1(\Sigma)$ *is free.*

Conjecture 4.19 includes all cases related to cluster algebras.

Noncommutative Skein Algebra. We use an extension of the usual noncommutative Jones skein algebra to allow marked points on **S** (but no punctures) as described by Muller (25). In that setting, $\mathrm{Sk}_q(\Sigma)$ also has a basis consisting of simple diagrams. The three bases generalize: for a simple loop L, define in $\mathrm{Sk}_q(\Sigma)$

$$\begin{aligned}\mathrm{Bang}^k(L) &= \langle L\rangle^k \\ \mathrm{Band}^k(L) &= U_k(\langle L\rangle) \\ \mathrm{Brac}^k(L) &= T_k(\langle L\rangle).\end{aligned}$$

Extend this to a complete basis as before. Note that U_k and T_k are the ordinary Chebyshev polynomials with integer coefficients.

We say that a basis for an algebra over $\mathbb{Z}[q^{\pm}]$ is *positive* if the structure constants for multiplication lie in $\mathbb{Z}_{\geq 0}[q^{\pm}]$.

Conjecture 4.20. *The bracelets basis is a positive basis for* $\mathrm{Sk}_q(\Sigma)$.

Evidence for *Conjecture 4.20* comes from $\mathrm{Sk}_q(T^2)$, where Frohman and Gelca computed the multiplication rules with respect to a suitable basis (24):

$$C_{a,b} \cdot C_{c,d} = q^{ad-bc}C_{a+c,b+d} + q^{-ad+bc}C_{a-c,b-d}. \qquad [23]$$

Conjecture 4.21. *For any connected surface* Σ *with nonempty boundary, the bands basis is a positive basis for* $\mathrm{Sk}_q(\Sigma)$.

Remark 4.22. *Although the bracelets basis is the subject of this paper, the bands basis has appeared in several different contexts, and arises naturally from representation theory. When we think of* $\mathrm{Sk}(\Sigma)$ *as functions on the twisted* $\mathrm{SL}_2(\mathbb{R})$ *character variety of* Σ, *the value of a single loop* L *with lifted holonomy* $\gamma \in \pi_1(UT\Sigma)$ is

$$\langle L\rangle = \pm\,\mathrm{tr}_2(\rho(\gamma)),$$

where ρ *is the* SL_2 *representation and* tr_2 *is the trace in the 2D representation of* SL_2. *With this setup, we have*

$$\begin{aligned}\mathrm{Band}^k(L) &= \pm\,\mathrm{tr}_{k+1}(\rho(\gamma)) \\ \mathrm{Brac}^k(L) &= \pm\,\mathrm{tr}_2(\rho(\gamma^k)).\end{aligned}$$

That is, for $\mathrm{Band}^k(L)$, *we take the trace in the k'th symmetric power of the defining representation of* SL_2 *(with dimension* $k+1$*), while for* $\mathrm{Brac}^k(L)$ *we take the trace of the* k*'th power of the loop*.

In the case of an annulus with two marked points (one on each boundary component), the skein algebra is contained in a quantized affine algebra, and Lusztig's dual canonical basis is the bands basis and not the bracelets basis (26).

5. Positivity

Definition 5.1. *A diagram D is (manifestly)* null *if it has a singular monogon based at a marked point. A crossing x of a diagram D is* weakly positive *if both resolutions of D at x are weakly positive or null, with at least one of the resolutions weakly positive; similarly for strongly positive.*

Here is the key lemma of the paper.

Lemma 5.2. *Let D be a weakly positive diagram that is not isotopic to a multibracelet. Then D has weakly positive crossing.*

Proof of Theorem 2, assuming Lemma 5.2: First suppose we have a weakly positive diagram D. We proceed by induction on the complexity of D (*Definition 3.5*).

If D is not isotopic to a multibracelet, apply *Lemma 5.2*: Since D has a weakly positive crossing, we have $\langle D\rangle = \langle D_1\rangle + \langle D_2\rangle$ where both D_i are weakly positive or null and strictly simpler than D. Null diagrams can be ignored (as they are 0 in the skein algebra). So, by induction, the $\langle D_i\rangle$ and therefore $\langle D\rangle$ have positive expansions in the bracelets basis.

If D is isotopic to a multibracelet D', it is nearly in the bracelets basis, except that it may have parallel bracelets or bracelets around punctures. Any bracelet around a puncture is equal to 2, so these may be removed. If D' has parallel bracelets, say $D' = \mathrm{Brac}^k(L)\mathrm{Brac}^l(L)D''$ for a simple loop L, then by *Proposition 4.4* and Eq. **7**,

$$\langle D\rangle = \langle D'\rangle = \langle D''\rangle\langle \mathrm{Brac}^{k+l}(L)\rangle + \langle D''\rangle\left\langle \mathrm{Brac}^{|k-l|}(L)\right\rangle. \quad [24]$$

The terms on the right have fewer parallel bracelets than D', and so we can repeat this reduction until we are left with a positive linear combination of elements of the bracelets basis.

If D is not positive, then by *Theorem 2.10* it can be reduced to a taut curve D' by a sequence of Reidemeister moves. Each move $D_i \to D_{i+1}$ either leaves the value in the skein algebra the same or multiplies it by an integer, as follows:

Move RO: $\langle D_i\rangle = -2\langle D_{i+1}\rangle$.

Move RI: $\langle D_i\rangle = -\langle D_{i+1}\rangle$.

Move RIb: $\langle D_i\rangle = 0$.

Moves RII, RIIb, and RIII: $\langle D_i\rangle = \langle D_{i+1}\rangle$.

Thus, $\langle D\rangle = k\langle D'\rangle$ for some integer k, and the theorem follows. □

Proof of Theorem 1, assuming Lemma 5.2: Given two diagrams D_1 and D_2 in the bracelets basis, find a taut representative D for $D_1 \cup D_2$ and apply *Theorem 2*. *Lemma 2.9* guarantees that $\langle D\rangle = \langle D_1\rangle\langle D_2\rangle$. □

So we only need to prove *Lemma 5.2*. We prove a slightly stronger version.

Lemma 5.3. *Let D be a taut diagram that is not a multibracelet. Then D is isotopic through RIII moves to a taut diagram D' that has a strongly positive crossing.*

Proof of Lemma 5.2, assuming Lemma 5.3: Let D be a weakly positive diagram. Then by assumption D can be reduced by regular isotopy to a taut diagram D', which by *Lemma 5.3* is regular isotopic to a diagram D'' with a strongly positive crossing x. By *Lemma 2.21*, x is also weakly positive. By *Lemma 2.12*, there is a crossing $m(x) \in \mathrm{Cross}(D)$ so that the resolutions of $m(x)$ are regular isotopic to the resolutions of x. Thus, $m(x)$ is also weakly positive. □

The plan to prove *Lemma 5.3* is to look for a crossing at the end of a *maximal bracelet chain*, a portion of the diagram that wraps around a simple loop (*Definition 5.6* below), like crossing x_3 in Fig. 5. This crossing will be strongly positive (*Lemma 5.9*). If there are no such crossings, each component of the diagram is a simple arc or a bracelet and any crossing between two components is strongly positive (*Lemma 5.10*).

First we build up some tools.

Lemma 5.4. *Suppose D is a taut diagram and $D \to D_1 + D_2$ is a crossing reduction with D_1 not strongly positive. Then there is a 0-gon or 1-gon $H \subset D_i$ passing through the reduction disk twice.*

Proof: By definition, if D_1 is not strongly positive it has a singular disk or monogon H. If H does not pass through the reduction disk, then it is also a singular monogon or disk for D, contradicting the assumption that D is taut. If H passes through the reduction disk once, then we can lift it to D: A 0-gon lifts to a 1-gon in D, and a 1-gon lifts to a 2-gon in D. However, a taut diagram cannot have any 1-gons, and by *Lemma 2.13* can only have 2-gons between two parallel strands. So H passes through the reduction disk at least twice. Since by definition the intervals making up a chain are disjoint, H must pass through the reduction disk exactly twice. □

Lemma 5.5. *Suppose D is a taut diagram, $x \in \mathrm{Cross}(D)$, and D_1 is the disconnected resolution of D at x (i.e., the resolution that increases the number of components). Then D_1 is strongly positive.*

Proof: By *Lemma 5.4*, if D_1 is not strongly positive there is a 0-gon or 1-gon passing through the reduction disk twice. However, a 0-gon or 1-gon cannot touch both components of the disconnected resolution, so this is impossible. □

Definition 5.6. *In a taut diagram D, recall that a* bracelet loop *is a component of D that is homotopic to $\mathrm{Brac}^k(L)$ for a simple loop L. A* bracelet chain *is a 1-chain H, in the sense of* Definition 2.7, *so that H° is homotopic to $\mathrm{Brac}^k(L)$ for some simple loop L and $k > 0$ and with H not a complete arc component of D. A* bracelet *is a bracelet loop or bracelet chain. A bracelet B has an* underlying loop *$L(B)$, around which it travels $n(B)$ times. Define $\gamma(B) \in \pi_1(\Sigma)$ to be the holonomy of $L(B)$, with basepoint and arc to the basepoint specified as necessary. The holonomy of B itself is $\gamma^n(B) := \gamma(B)^{n(B)}$. Finally, a* maximal bracelet *is a bracelet B for which there is no other bracelet B' with $L(B') = L(B)$ and $n(B') > n(B)$.*

Lemma 5.7. *Let D be a connected taut diagram with at least one self-intersection. Then D has a maximal bracelet.*

Proof: Among all 1-chains H of D with corners at crossings, pick one that is minimal with respect to inclusion. (There is one since D has a self-intersection.) Then H° is necessarily a simple loop, so H is a bracelet. Now take B to be a maximal bracelet with $L(B) = H^\circ$. □

We can arrange for maximal bracelets to lie in a good position.

Lemma 5.8. *Let D be a taut diagram and let $B \subset D$ be a bracelet chain. Then (B, D) is isotopic through RIII moves to (B', D') where D' and $(B')^\circ$ are both taut.*

Proof: Suppose that B° is not taut. We will reduce the number of self-intersections of B° by RIII moves on D.

Let x be the intersection at the end of B and let $\bar{x}$ be a point on B° near x. Note that B° is part of the disconnected resolution of D at x, so by *Lemma 5.5* it is strongly positive. Thus, by *Theorem 2.17*, B° has a singular 2-gon G. Since D is taut, G must pass over $\bar{x}$; let $G = (S_1, S_2)$, with the segment S_1 containing $\bar{x}$. Pick G so that S_1 is minimal with respect to inclusion.

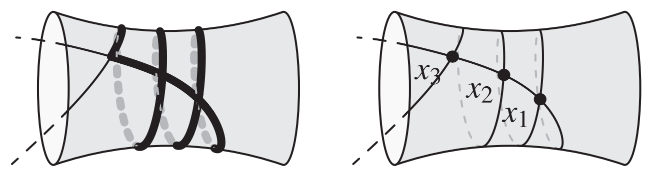


Fig. 5. A maximal bracelet chain and the labeling of the self-intersections.

Since G is null-homotopic, it lifts to a 2-gon $\tilde{G} = (\tilde{S}_1, \tilde{S}_2)$ in the universal cover $\tilde{\Sigma}$ of Σ. Since S_1 is minimal, $\tilde{S}_1$ and $\tilde{S}_2$ do not intersect, so $\tilde{G}$ is an embedded 2-gon. Let $\tilde{A}$ be the intersection of the preimage of D with the disk bounded by $\tilde{G}$. Because D is taut, no component of $\tilde{A}$ can meet $\tilde{S}_2$ twice, and because S_1 is minimal, no component of $\tilde{A}$ can meet $\tilde{S}_1$ twice. So each component of $\tilde{A}$ is an arc running from $\tilde{S}_1$ to $\tilde{S}_2$. Now let D' be D, but with S_2 pushed over G to run parallel to S_1 outside of G, and let B' be the bracelet in D' with corner at x. Then D' has the same number of self-intersections as D, but $(B')^\circ$ has fewer self-intersections, as desired.

The homotopy from S_2 to a parallel copy of S_1 can be done using only RIII moves, since it does not change the intersection number (lemma 1.6 in ref. 19). □

Following *Lemma 5.8*, we say that a bracelet chain $B \subset D$ is *taut* if D and B° are both taut. In this case, we follow *Proposition 2.19* and label the self-intersection of B that cuts off holonomy $\gamma(B)^k$ by $x_k(B)$ for $k = 1, \ldots, n$, as in Fig. 5. (To define the holonomy, pick a basepoint on B and connect the intersection to the basepoint by traveling along B.)

Lemma 5.9. *Let $B \subset D$ be a taut maximal bracelet that is a chain. Then for $k > n(B)/2$, the crossing $x_k(B)$ is strongly positive.*

Proof: For convenience, we assume that D has only one connected component. Suppose D' is a resolution of D at $x_k(B)$ that is not strongly positive. By *Lemma 5.5*, D' is the connected resolution. Let B' be the subbracelet of B cut off by $x_k(B)$. Let H be the 0-gon from *Lemma 5.4*. If H is a 1-gon, its endpoints are either both in B' or both in $D \backslash B'$. We write γ for $\gamma(B)$ and proceed by cases on H.

- If H is a 0-gon, then D is a loop with holonomy γ^{2k}.
- If H is a 1-gon with endpoints in B', the endpoints are at $x_l(B)$ for some $l < k$. Let the holonomy of $D \backslash B'$ (as a 1-chain) be ρ. Then the holonomy of H is $\rho^{-1}\gamma^{k-l} = 1$, which implies that D is a loop with holonomy $\rho\gamma^k = \gamma^{2k-l}$.
- If H is a 1-gon with endpoints in $D \backslash B'$, let y be the corresponding corner. There is a corresponding 1-chain C in D with corner at y. Let ρ be the holonomy of $C \backslash B'$. Then the holonomy of H° is $\rho \cdot \gamma^{-k} = 1$ so $\rho = \gamma^k$, and the holonomy of C is γ^{2k}.

In almost all cases, we found a bracelet chain which contradicts the maximality of B. The only remaining case is when $k = (n+1)/2$ and B is contained in an arc with holonomy conjugate to γ^{n+1}. In this case, the connected resolution is null. The disconnected resolution is strongly positive by *Lemma 5.5*, so again the crossing is strongly positive. □

Lemma 5.10. *Let $D_1 \cup D_2$ be a taut diagram where D_1 and D_2 are simple arcs or bracelet loops. Then any crossing between D_1 and D_2 is strongly positive.*

Proof: Let x be a crossing between D_1 and D_2, let D' be a resolution of $D_1 \cup D_2$ at x, and suppose H is a 0-gon or 1-gon of D'. By *Lemma 5.4*, H must pass through the reduction disk at x twice, which means that one of the curves (say, D_2) must be a bracelet loop and H must run completely over D_2, with both endpoints in D_1.

If D_1 is an arc, then (since D_1 is simple) the endpoints of H must be the endpoints of D_1, which are necessarily at the same marked point. Let the holonomy of $(D_1)^\circ$ be $\rho \in \pi_1(\Sigma, x)$. If ρ is a power of $\gamma(D_2)$, D_1 and D_2 do not intersect. However, otherwise the holonomy of H° is $\rho \cdot \gamma^{\pm n}(D_2)$, which is not 1 by *Lemma 2.22*.

If D_1 and D_2 are both bracelet loops, again $\gamma(D_1) \neq \gamma(D_2)$ in $\pi_1(\Sigma, x)$, since otherwise D_1 and D_2 would not intersect. The holonomy of H° is $\gamma(D_1)^k \cdot \gamma(D_2)^{n(D_2)}$ for some $1 \le k \le n(D_1)$. Again this cannot be 1 by *Lemma 2.22*. □

Proof of Lemma 5.3: If any component C of D is not a simple arc or bracelet loop, then by *Lemma 5.7* it has a maximal bracelet chain B, which we can assume (after RIII moves) to be taut by *Lemma 5.8*. Then $x_n(B)$ is strongly positive by *Lemma 5.9*.

If all components of D are simple arcs or bracelets loops, by *Lemma 5.10*, any crossing between different components of D is strongly positive.

Thus, if D is not a multibracelet, we have exhibited a strongly positive crossing in a diagram isotopic to D through RIII moves. □

Remark 5.11. *A closer analysis of the proof shows that every taut diagram D has a strongly positive crossing (i.e., we do not need to do an isotopy first): A crossing x near the end of a maximal bracelet chain B is strongly positive, whether or not B is taut. Essentially, the RIII moves to make B taut (following Lemma 5.8) are also RIII moves on the connected resolution of D at x.*

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