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Debreu's social equilibrium existence theorem

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Game theory is the study of strategic interaction: how a person should and will behave if her actions affect others and their actions affect her. (We use the term "person" here, but "agent" might be more appropriate, because, in the literature, game theory applies to firms, nations, or even genes, as well as to individual people.) Among fields of application, the theory has had its biggest influence on economics (but has also had considerable sway over political science, evolutionary biology, and computer science, among other disciplines).

Economists are interested in predicting (or at least explaining) behavior in economic environments, for example, what products firms will offer for sale, which goods and services households will purchase, or what securities (e.g., stocks or bonds) investors will trade. To see how game theory has contributed to this effort—and, in fact, has revolutionized economics—think about trying to analyze the behavior of firms in the American automobile industry. To simplify, let's suppose that the industry is dominated by three companies, General Motors, Ford, and Chrysler (these three did indeed dominate for many years). Now, for General Motors to figure out what models to launch, how many units of each model to produce, and what prices to set for them, it has to anticipate what Ford and Chrysler will be doing. However, the same sort of anticipation is required of Ford and Chrysler. That is, Ford's best course of action will depend on what it thinks GM and Chrysler are going to do. Actually, it's even more complex than that, because GM will understand that Ford is trying to forecast GM's behavior. Therefore, to predict Ford's actions, GM also has to anticipate Ford's anticipation, and so on, to higher and higher levels of prediction. Such mutual forecasting is clearly crucial to an analysis of the automobile market but is potentially extremely complex.

Nash's Contribution

John Nash (1) proposed a simple but powerful concept that cut through this complexity. Because each player in the game (each firm in the automobile industry) is trying to optimize given its anticipation of others' strategies, he defined an "equilibrium point" (now called a Nash equilibrium) as a fixed point of these optimizations. Specifically, a Nash equilibrium is a configuration of strategies—one for each player—such that no player gains by unilaterally changing its strategy. The concept enormously simplifies the forecasting problem: If each player anticipates that others will use their Nash equilibrium strategies, then it will do so too, and the forecasts will be self-confirming. The success of the Nash equilibrium concept as a predictive and explanatory tool in such settings as the automobile industry transformed much of economics.

Nash equilibrium also helps illuminate behavior in vast array of environments outside economics. To take a simple example, consider how you behaved if you drove to work this morning. If you are in the United States, you probably drove on the right-hand side of the road. Why? Because you anticipated that oncoming drivers would stay to the right, and they anticipated the same of you. These anticipations were self-confirming because everybody driving on the right constitutes a Nash equilibrium; switching to another strategy unilaterally could well be disastrous.

More formally, suppose that there are n players, where each player i $(i = 1, ..., n)$ has a strategy space S_i, a set of possible behaviors. In the driving game, for instance, S_i consists of "driving on the right" and "driving on the left." A game is a function

$$g : S_1 \times \ldots \times S_n \to \mathbb{R}^n, \quad [1]$$

where $g(s_1, \ldots, s_n) = (g_1(s_1, \ldots, s_n), \ldots g_n(s_1, \ldots, s_n))$, and $g_i(s_1, \ldots, s_n)$ is player i's payoff when strategies $(s_1, \ldots, s_n)$ are played [when focusing on player i's payoff, we will sometimes write $(s_1, \ldots, s_n)$ as (s_i, s_{-i}), where s_{-i} is the $(n-1)$-tuple of the other players' strategies]. A Nash equilibrium is an n-tuple of strategies $(s_1^*, \ldots, s_n^*) = (s_i^*, s_{-i}^*)$ such that

$$g_i(s_1^*, \ldots, s_n^*) \ge g_i(s_i, s_{-i}^*), \quad i = 1, \ldots, n,$$

$$\text{and all } s_i \in S_i. \quad [2]$$

In words, no player i gains by unilaterally deviating from s_i^* to s_i.

Nash (2) showed that a finite game (a game in which each player's strategy space is finite) has a Nash equilibrium, once players are allowed to randomize over their strategies. Formally, a mixed or randomized strategy for player i is a probability distribution p_i over player i's pure strategies S_i, where, for each $s_i \in S_i$, $p_i(s_i)$ is the probability that player i uses strategy s_i. If players use mixed strategies $(p_1, \ldots, p_n)$, then, assuming these randomizations are independent across players,

$$\text{player } i\text{'s expected payoff} = \sum_{s_1 \in S_1} \cdots \sum_{s_n \in S_n} g_i(s_1, \ldots, s_n)\, p_1(s_1) \ldots p_n(s_n)$$

$$= g_i(p_1, \ldots, p_n) = g_i(p_i, p_{-i}).$$

Nash proved that there exists an n-tuple $(p_1^*, \ldots, p_n^*)$ such that

$$g_i(p_1^*, \ldots, p_n^*) \ge g_i(p_i, p_{-i}^*) \quad \text{for all } i \text{ and } p_i. \quad [3]$$

Nash's proof was short and simple. For each $(n-1)$-tuple p_{-i} of other players' mixed strategies, let $\psi_i(p_{-i}) = \{p_i | g_i(p_i, p_{-i}) \ge g_i(p_i', p_{-i})$ for all $p_i'\}$. The mapping ψ_i is player i's "best-response" correspondence: $\psi_i(p_{-i})$ consists of all randomized strategies that maximize her payoff given p_{-i}. Let ψ be the cross product of the ψ_is: $\psi(p_1, \ldots, p_n) = \psi_1(p_{-1}) \times \ldots \times \psi_n(p_{-n})$. Then, ψ is an upper hemicontinuous, convex-valued, and non-empty-valued correspondence that maps the set of

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randomized strategy n-tuples to itself. Thus, from the Kakutani fixed point theorem (3), a fixed point $(p_1^*, \ldots, p_n^*) \in \psi(p_1^*, \ldots, p_n^*)$ exists, and, by definition of ψ, $(p_1^*, \ldots, p_n^*)$ satisfies Formula **3**, i.e., it is a Nash equilibrium.

A serious limitation of Formula **1** is that it assumes that a player's set of options is not affected by what others do; that is, player i's strategy space S_i does not depend on other players' strategies. This assumption, admittedly, fits many settings. In the driving game, for example, you are free to drive on the right or left independently of others' choices (although the consequences of which side you drive on will certainly depend on those choices). However, there are many other games in which the assumption does not make sense. Imagine, for instance, a supermarket shopper interested in buying the basket of goods that is best for her family. Naturally, she can't choose any basket she pleases; she must stay within her budget. However, this budget is not independent of what other players in the economy do; it will presumably depend, for example, on what her employer pays her. Also, given her budget, which baskets are affordable will depend on the various goods' prices, which, in turn, are determined by supply and demand in the economy.

Debreu's Contribution

This is where Gérard Debreu's "A social equilibrium existence theorem" (4) comes in. Debreu (4) extended the very notion of a game to allow for the possibility that a player's set of strategies might depend on the strategy choices of other players. Specifically, he supposed that for each player i and each $(n-1)$-tuple of other players' strategies s_{-i}, player i's choices are limited to some subset $\gamma_i(s_{-i}) \subseteq S_i$; that is, $\gamma_i(s_{-i})$ are the strategies that are feasible for player i when others play s_{-i}. In this more general setting, a social equilibrium is an n-tuple $(s_1^*, \ldots, s_n^*)$ such that, for all i,

$$s_i^* \in \gamma_i(s_{-i}^*) \qquad [4]$$

and

$$g_i(s_1^*, \ldots, s_n^*) \geq g_i(s_i, s_{-i}^*) \quad \text{for all } s_i \in \gamma_i(s_{-i}^*). \qquad [5]$$

In words, each player, in equilibrium, must be playing a feasible strategy (Formula **4**), and no player can gain by unilaterally deviating to some other feasible strategy (Formula **5**). (Because Debreu was interested in equilibrium in pure strategies, we have not introduced randomizations in Formula **5**.)

Debreu (4) also gave conditions under which a social equilibrium (in pure strategies) is guaranteed to exist. Specifically, he showed that if, for all i, $(i) S_i$ is a compact and convex subset of Euclidean space, (ii) g_i is continuous in (s_i, s_{-i}) and quasi-concave in s_i, i.e., $g_i(\lambda s_i + (1-\lambda)s_i', s_{-i}) \geq \min\{g_i(s_i, s_{-i}), \; g_i(s_i', s_{-i})\}$ for all λ with $0 \leq \lambda \leq 1$, and (iii) γ_i is upper and lower hemicontinuous, convex-valued, and non-empty-valued, then there exists an n-tuple $(s_1^*, \ldots, s_n^*)$ satisfying Formulas **4** and **5**.

Because Nash (2) was working with randomized strategies, he obtained conditions i and ii automatically: The set of randomizations over a finite strategy set is a compact and convex subset of Euclidean space, and the corresponding payoff function $g_i(p_i, p_{-i})$ is continuous in (p_i, p_{-i}) and linear (and hence quasi-concave) in p_i. Thus Debreu's use of conditions i and ii in his proof of existence was a straightforward generalization of Nash. The novel part of Debreu's argument was his use of the lower hemicontinuity of γ_i to show that best-response correspondences are upper hemicontinuous. For all i and s_{-i}, let

$$\hat{\psi}_i(s_{-i}) = \{s_i \in \gamma_i(s_{-i}) \,|\, g_i(s_i, s_{-i}) \geq g_i(s_i', s_{-i}) \text{ for all } s_i' \in \gamma_i(s_{-i})\}.$$

The correspondence $\hat{\psi}_i$ selects player i's feasible best responses to s_{-i}. $\hat{\psi}_i$ is non-empty-valued because S_i is compact, g_i is continuous, and γ_i is upper hemicontinuous and non-empty-valued; it is convex-valued because g_i is quasi-concave and γ_i is convex-valued [so far, the argument parallels Nash (2)]. To see that $\hat{\psi}_i$ is upper hemicontinuous, consider a sequence $(s_i^m, s_{-i}^m) \to (s_i, s_{-i})$ where, for all m, $s_i^m \in \hat{\psi}_i(s_{-i}^m)$. Now, $s_i \in \gamma_i(s_{-i})$ from the upper hemicontinuity of γ_i. If $s_i \notin \hat{\psi}_i(s_{-i})$, there must exist $\hat{s}_i \in \gamma_i(s_{-i})$ such that $g_i(\hat{s}_i, s_{-i}) > g_i(s_i, s_{-i})$. However, because γ_i is lower hemicontinuous, there must exist $\hat{s}_i^m \to \hat{s}_{-i}$ such that $\hat{s}_i^m \in \hat{\gamma}_i(s_{-i}^m)$ for all m. Hence, by continuity of g_i, $g_i(\hat{s}_i, s_{-i}^m) > g_i(\hat{s}_i^m, s_{-i}^m)$ for m big enough, a contradiction of $s_i^m \in \hat{\psi}_i(s_{-i}^m)$. We conclude that $s_i \in \hat{\psi}_i(s_{-i})$. Thus, Debreu concludes that $\hat{\psi}_i$ is upper hemicontinuous, and the Kakutani theorem now applies to give the existence result.

One centrally important application of Debreu's theorem was to establish the existence of a general equilibrium in a competitive economy. An economy is competitive if the players—the producers and consumers—are too small as individuals to affect market prices. In a general equilibrium of such an economy, the profit-maximizing choices of producers and the preference-maximizing choices of consumers (given equilibrium prices) are consistent in the sense that supply equals demand in every market. Arrow and Debreu (5) laid out the model of a competitive economy and used Debreu's theorem to show that a general equilibrium exists in this model. The Debreu theorem was of crucial importance here because consumers' feasible choices in the Arrow–Debreu model depend on their incomes and on prices, as in the supermarket example above.

Debreu (4) succeeded in (i) generalizing the standard framework of game theory by allowing players' feasible strategy spaces to depend on others' behavior, (ii) formulating conditions under which a social equilibrium (the natural extension of Nash equilibrium) exists in a generalized game, and (iii) providing the key tool for Arrow and Debreu (5) to establish the existence of a general equilibrium in a competitive economy. For all these reasons, it is a landmark paper.

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1 Nash JF (1950) Equilibrium points in n-person games. *Proc Natl Acad Sci USA* 36(1):48–49.
2 Nash J (1951) Non-cooperative games. *Ann Math* 54(2):289–295.
3 Kakutani S (1941) A generalization of Brouwer's fixed point theorem. *Duke Math J* 8(3):457–459.
4 Debreu G (1952) A social equilibrium existence theorem. *Proc Natl Acad Sci USA* 38(10):886–893.
5 Arrow K, Debreu G (1954) Existence of an equilibrium for a competitive economy. *Econometrica* 22(3): 265–290.