Real space renormalization group theory of disordered models of glasses

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We develop a real space renormalization group analysis of disordered models of glasses, in particular of the spin models at the origin of the random first-order transition theory. We find three fixed points, respectively, associated with the liquid state, with the critical behavior, and with the glass state. The latter two are zero-temperature ones; this provides a natural explanation of the growth of effective activation energy scale and the concomitant huge increase of relaxation time approaching the glass transition. The lower critical dimension depends on the nature of the interacting degrees of freedom and is higher than three for all models. This does not prevent 3D systems from being glassy. Indeed, we find that their renormalization group flow is affected by the fixed points existing in higher dimension and in consequence is nontrivial. Within our theoretical framework, the glass transition results in an avoided phase transition.

Understanding the glass transition and the properties of glasses is still an open problem, despite many years of research and a lot of recent progress (1, 2). The glass state poses to statistical physicists a conundrum and a challenge. It is a state of matter that is rigid like a crystal, but disordered like a liquid. A glass is a solid that forms progressively from a supercooled liquid in a way strongly suggesting the existence of an underlying critical phenomenon. In fact, the relaxation time scale and the viscosity of supercooled liquid increase very fast when lowering the temperature, indicating a growing effective energy scale, as would be expected at a phase transition. If such a phase transition really exists (or at least is weakly avoided), then it certainly has to be of a new kind because the corresponding phenomenology does not fit in any standard paradigm. The challenge to understand glasses and the glass transition is that one has to deal with a system of many interacting degrees of freedom characterized by a very rugged energy landscape. This is a fascinating problem that has ramifications in contexts as different as computer science and biology, as well as in other branches of physics. Standard tools of statistical mechanics are not well adapted to tackle it. In fact, part of the current research is devoted to developing new theoretical methods and concepts able to deal with it (3).

Among the several different theoretical approaches aimed at explaining the glass transition, the Random First Order Transition (RFOT) theory, originally introduced by Kirkpatrick, Thirumalai, and Wolynes (KTW), stands as one of the most prominent candidates (4–6). In a series of pioneering papers, KTW investigated the potential relationships between supercooled liquids and mean-field generalized spin glasses (such as the fully connected Potts glass and random p-spin models) (7–9). These and other results (10) allowed KTW to introduce in ref. 4 a finite dimensional density functional theory for the glass transition that tied together in a coherent scenario many elements of the description of supercooled liquids. What at that time was considered by some a great insight (but also an absurd assumption by others) proved to be right in recent years. It is now clear that the mean-field theory of glasses, defined in terms of the solution in the limit of large spatial dimension d, is the same as the generalized spin-glass models initially considered by KTW. This result comes from a long series of works that culminated in the exact solution of hard spheres in the infinite dimensional limits (11–13). The solution of models with smooth potentials at finite temperature was also completed recently (14). Going beyond d = ∞ results is a quite challenging task, and it is at the core of current research: A theoretical approach for finite dimensional systems was originally developed by KTW (4), and in recent years, theoretical and numerical results have further developed these ideas (15–18). What makes going beyond mean-field theory hard is that the growth of time and length scales approaching the glass transition is related to nonperturbative effects in 1/d. Unlike in standard critical phenomena, where mean-field theory is quantitatively correct above the upper critical dimension and qualitatively correct below, for glasses, mean-field theory has many caveats, even in large dimensions, the most problematic being the infinite lifetime of metastable amorphous states (barriers diverge exponentially in d, hence are strictly infinite for d = ∞). The very process characterizing slow and glassy dynamics—the evolution from one metastable state to another—is absent in mean-field theory, which therefore cannot explain by itself the growth of time and length scale approaching the glass transition.

In standard critical phenomena, the renormalization group (RG) proved to be the theoretical framework that allows one to go beyond the mean-field approximation and obtain a complete theory. In the case of systems characterized by a rugged energy landscape, such a method is still far from being applicable in general. There are both technical and conceptual difficulties. On the one hand, it is necessary to develop an RG procedure able to go beyond the mean-field approximation and obtain a complete theory. In the case of systems characterized by a rugged energy landscape, such a method is still far from being applicable in general. There are both technical and conceptual difficulties. On the one hand, it is necessary to develop an RG procedure able to deal with such nonperturbative and nonanalytic effects.

Significance

The hallmark of the glass transition is the very fast increase of the relaxation time τ ∼ log T. Experimental results have rationalized this fact in terms of a growing effective activation energy scale, ∆ = T log τ , characterizing the height of the barriers in the rough energy landscape of glass-forming liquids. By applying the renormalization group method to disordered models of glasses, we show that barriers related to degrees of freedom correlated on the scale ξ ∼ ℓ θ , where θ is a critical exponent. The growth of the static length-scale ξ, which is the typical size of the statically correlated regions, thus leads to ξ ∼ τ θ and provides a general explanation for glassy behavior.

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to take into account the complexity of the rugged landscape: One cannot follow the flow of a few coupling constants, but instead the RG has to be functional and in many cases nonperturbative. On the other hand, fixed points (FPs) describing the critical behavior and the low-temperature phase are possibly of new kinds and require a conceptual leap forward. In some cases, such as Random Manifolds (19, 20) and the Random Field Ising Model (RFIM) (21), these difficulties were overcome and a complete theory was worked out; in others, such as glasses, the state of the art is less advanced, despite some recent progress (22–25).

With the aim of developing a RG analysis of the glass transition, in this work, we present a complete real-space RG analysis of the class of disordered models from which RFOT originated. Because at least within mean-field theory, spins and particles models are in the same universality class, it is conceivable that our findings could hold for particle systems, too.

Before summarizing our main results, it is useful to discuss in general the kind of RG flow one expects for a system undergoing a glass transition. Several ideas were already put forward in the literature (4, 25–28). In the following, we will focus on the thermodynamics and infer how this influences dynamics. This perspective is rooted in the RFOT theory and also in other approaches (5, 29); it received positive, but not conclusive, confirmations recently (see refs. 30 and 31 and references therein).

The very basic experimental fact about systems approaching the glass transition is the dramatic increase of the relaxation time \( \tau \). The effective energy barriers scale (henceforth denoted \( \Delta \)) increases, possibly as a diverging power law in the distance from the transition, and leads to a growth of \( \tau \), which is much faster than the usual power law of standard critical phenomena. This strongly suggests that the critical point associated with the glass transition (if it exists) should be associated with a zero-temperature FP* of the RG (28, 32). In fact, in this case, the thermodynamic energy scale \( \Delta(\ell) \), and consequently the energy scale of barriers between different states, should scale as \( \Delta(\ell) \sim \Delta_0 \ell^d \), where \( \ell \) is the scale over which fluctuations have been integrated out by RG. Close to the critical point, under renormalization, the system is characterized by a larger and larger \( \Delta \) until the static correlation length \( \xi \) is reached. On scales larger than \( \xi \), the system can be considered as an ensemble of weakly interacting subparts, each one characterized by the energy scale \( \Delta(\xi) \), that could be considered as the energy scale to spend for a cooperative rearrangement of such a subpart. The time scale for relaxation is therefore given by the Arrhenius law applied to each subpart. This leads to \( \tau \approx \tau_0 e^{\Delta(\ell)/T} = \tau_0 e^{\Delta_0 \ell^d/T} \), which is the law conjectured to hold (for different reasons) in the different thermodynamic theories of the glass transition (5, 29).

This gedanken description of the RG flow for glasses is simplified in many ways. Among the several possible additional phenomena that could have been taken into account, three are particularly important. First, as discussed in the introduction, rugged landscapes are complex objects that cannot be simply described by just one coupling constant; one has at least to consider that \( \Delta \) fluctuates in space, so what really flows is the distribution \( P(\Delta) \). As a consequence, in the discussion above, \( \Delta \) should be considered the typical value of the energy scale extracted from \( P(\Delta) \). This difficulty is related to an important physical ingredient: Glassy systems are disordered because they contain either quenched or self-induced disorder. Disorder leads naturally to fluctuations of the energy scales and also to a competition between the energy scale characterizing the interaction between the degrees of freedom, \( \Delta \), and an idiosyncratic energy scale, that we shall denote \( h \), favoring specific local configurations (in the case of the RFIM, the former is associated with the ferromagnetic coupling and the latter with the random field). Both the distribution of \( \Delta \) and \( h \) have to be taken into account, because they clearly compete with each other. The idiosyncratic disorder \( h \) could even preclude the growth of \( \Delta \) and, hence, wipe out the transition. As we shall find, the critical point discussed above might be avoided but still be meaningful. A way to induce avoided FPs was discussed by Kivelson, Tarjus, and colleagues in the context of frustration limited domain theory (29) and was based on the idea that the transition, although absent in three dimensions, exists in curved space. Another possibility is that the transition exists in dimensions higher than three only, but nevertheless influences the 3D behavior. Because of the regularity of the RG equations, which generically do not show chaotic irregular behavior, the RG flow changes smoothly by varying the dimension. As a consequence, the RG flow below the lower critical dimension, \( d_L \), can be very much influenced by the one at \( d_L \), leading to avoided critical behavior. This is a conceivable possibility for the RFOT, which should exist in high enough dimension (5, 11–13).

What we find in our analysis contains all of the ingredients discussed above. At high enough dimension, higher than three, disordered models of glasses are characterized by a zero temperature critical FP, \( CR_{FP} \), that emerges from the competition between \( \Delta \) and \( h \) and governs the critical behavior associated to the glass transition. We also find a nontrivial FP, \( G_{FP} \), describing the low-temperature phase (i.e., the amorphous solid state), to which the system flows on large length scales for \( T < T_c \).

The qualitative RG diagram and the two fixed points are shown in Fig. 1. Some models, characterized by a too-strong idiosyncratic disorder, lie in RG sense far from the basin of attraction of \( G_{FP} \), and, hence, they do not display glass behavior in finite dimension. Others are instead characterized by a bona fide glass transition. By studying the dependence of \( CR_{FP} \) and \( G_{FP} \) on dimensionality, our results indicate that both FPs simultaneously disappear without coalescing for a dimension \( d_{CR} \) higher than three. Note that the larger the number of degrees of freedom strongly interacting on small length scale, the lower \( d_{CR} \) and the closer approaches three, but never reaches it, as shown in Fig. 2 Inset. By considering noninteger dimensions, as is usually done in RG study, we find that when the number of degrees of freedom becomes very large, \( d_{CR} \rightarrow 1.8 \). The behavior below \( d_{CR} \) is very similar to that at \( d_{CR} \); it departs from it only when approaching the vestige of \( CR_{FP} \) and \( G_{FP} \) (the region in couplings space where \( CR_{FP} \) and \( G_{FP} \) lie for \( d = d_{CR} \)). This happens on larger and larger length scale the more \( d \) is close to \( d_{CR} \). In fact, by analyzing the RG flow in three dimensions, we find that, although the glass transition is avoided, the existence of FPs at \( d_{CR} \) affects the flow for \( d = 3 \) and leads to glassy behavior and avoided criticality.

**Analyzed Models**

Our analysis applies to generic spin or Potts-disordered finite-range models defined on cubic (or hypercubic) lattices. Because the interaction range is finite (i.e., degrees of freedom distant more than \( \ell_t \) do not interact), we divide the lattice in a sublattice of blocks of size \( \ell_t^3 \). The degrees of freedom inside one block can be “packed” into one single degree of freedom \( \sigma \), that takes \( q = 2^M \) values (where, e.g., \( M = \ell_t^d \) for spin variables). In terms of these new variables, \( \sigma_i \), we have now a model with interaction between closest neighbors only. Without loss of generality, we consider only pairwise interactions between the \( \sigma_i \) of
the closest blocks. We, therefore, end up with a very generic set of models that we will call the $M$–value models, which are nearest-neighbor spin-disordered models, for which “spins” can take $q = 2^M$ values and with a Hamiltonian of the form:

$$H(\{\sigma\}) = \sum_{(ij)} E_{ij}(\sigma_i, \sigma_j),$$

where $E_{ij}(\sigma_i, \sigma_j)$ are a set of random link energies. Depending on their choice, one can realize a large variety of models. In particular, all disordered spin and Potts systems known to have an RFOT transition, and thus glassy features, within the mean-field approximation belong to this general class. The RG analysis of the $M$-value models thus allows us to draw very general conclusions. Different models characterized by the same value of $M$ only correspond to different initial conditions for the RG flow, but the FPs they can possibly reach are the same; they only depend on the value of $M$. The models we have focused on in this work are the Finite Dimensional Random Energy Models (REMs) that correspond to choosing all $E_{ij}(\sigma, \tau)$ in Eq. 1 as independent identically distributed (i.i.d.) Gaussian random variables extracted with zero mean and variance $M/2$ (33–35); $M - p$ spin-disordered models, which are finite version of the $p$-spin-disordered models that have been extensively studied as mean-field models of glasses (36, 37), and the Super-Potts models we showed recently to disordered Potts model recently introduced (41). Our analysis can generalize to the closest blocks. We neglect them for simplicity. In any case, they become negligible with respect to the pairwise interaction between nearest neighbor blocks if the size of the blocks is taken much larger than $l_i$.

To gain some general intuition about the model defined in Eq. 1, we let us decompose the couplings in the following way:

$$E(\sigma_1, \sigma_2) = J(\sigma_1 - \sigma_2) - H_L(\sigma_1) - H_R(\sigma_2) + C.$$  

where $J$ and $H$ and $C$ can be interpreted, respectively, as the true interactions between degrees of freedom, the external fields, and a constant. They can be fixed univocally from $E(\sigma_1, \sigma_2)$ by using the following set of relations. $C$ is naturally defined via the equation $C = \frac{1}{2} \sum_{\sigma_1, \sigma_2} E(\sigma_1, \sigma_2)$. The relation verified by the external field contribution on $\sigma_1$ ($\sigma_2$) is $\sum_{\sigma_2} H_L(\sigma_1) = 0$ ($\sum_{\sigma_1} H_R(\sigma_2) = 0$). It encodes the fact that the field should favor independently of the value of $\sigma_2$; hence, by summing over $\sigma_1$, one gets a constant, which has to be taken equal to zero to avoid double-counting with $C$. Finally, the interaction is what remains of $E(\sigma_1, \sigma_2)$ when the field and the constant are subtracted off and is defined via the relation $\sum_{\sigma_2} J(\sigma_1, \sigma_2) = \sum_{\sigma_1} J(\sigma_1, \sigma_2) = 0$. The explicit equations for $J$, and $H$ are reported in SI Text, RG Procedure. It is easy to verify that, for standard spin models, $J$ and $H$ correspond to the usual definition of magnetic interaction and magnetic field. Note that even if some of those terms are not present in the original model, they are generated by the RG flow. The decomposition performed in Eq. 2 makes clear that the models we are considering can be thought of as a generalized version of random magnets with pairwise interactions. A natural method to perform a real-space RG procedure is therefore using the Migdal–Kadanoff (MK) approximation (42, 43). This was proven to be able to capture nonperturbative zero-temperature FPs, such as the ones conjectured to play a role in the glass transition, and therefore it is a very good candidate to perform an RG analysis. In the case of spin-glasses, MK-RG provides a basis for the droplet model and correctly finds that the critical point is standard, whereas the low-temperature FP is nontrivial (44). In the case of the RFIM, it correctly finds that the critical behavior is governed by a zero-temperature FP, whereas the low-temperature phase is trivially ferromagnetic (45). MK-RG has other advantages: In particular, it is exact in one dimension and often provides a good quantitative approximation for the values of the critical exponents in not-too-high dimensions. Needless to say, it also suffers from some important drawbacks: In particular, it does not reveal the existence of an upper critical dimension (when this exists) and does not allow one to make a direct connection with mean-field theory. All in all, it has proven to provide valuable guidelines for the behavior of finite dimensional systems. It has a predictive power similar to the one of mean-field theory for phase diagrams: It provides a qualitatively correct description of RG flows, very often describing correctly the nature of the FPs, but being unable to provide accurate results, except in low dimensions. In the following, we present the analysis of the RG flow of the very broad class of disordered models of glasses discussed previously. There are several variants of the MK-RG. We use the one that turns out to be exact on hierarchical lattices (HLs). A HL is generated iteratively (Fig. 2 Inset and Fig. S2) at the step $G = 0$ two spins are connected by a single link. At each step $G$, each link is replaced by $b$ parallel branches, made of $2b$ bonds. The effective dimension of this model is $d = 1 + \ln(b)/\ln(2)$ (4d-45). The MK-RG, which is exact on HL and only approximate for hypercubic lattices in finite dimensions, consists of integrating out progressively degrees of freedom on larger and larger scales as shown in Fig. S1. It proceeds by retracing back the construction of the HL and iteratively summing over the $\sigma$ on each branch. In this way, after one RG step, we get a renormalized $M$-value model whose unit of length is doubled, with new renormalized energies. We thus follow the RG flow for the probability distribution of the renormalized energy over larger and larger length-scales. For the detailed procedure, we refer to SI Text, RG Procedure.
at \((v_H/v_J, J/v_J) = (\infty, \infty)\), one finds two different FPs, corresponding to the critical regime \((CR_{FP})\) and to the low-temperature glass phase \((G_{FP})\), respectively. Different models correspond to different initial conditions for the flow, as shown by the two dashed lines in Fig. 1. Accordingly, the value of the critical temperature \(T_c\) at which the flow lies in the basin of attraction of \(CR_{FP}\) is different. If the disidosyncratic disorder is too strong initially (i.e., \(v_H/v_J\) is too large), no transition takes place. For all three models we analyzed, the initial ratio is low enough to induce a transition. Note that even if we put artificially \(v_H = 0\) at the first step, the external field is generated under RG.\(^1\) The FPs, \(CR_{FP}\) and \(G_{FP}\), are zero-temperature ones: Both \(v_J\) and \(v_H\) increase with \(\ell = 2^d\) as \(v_J(x) \propto \ell^\theta, v_H(x) \propto \ell^\theta\) in such a way that \(v_H(x)/v_J(x) \rightarrow \text{constant} = (v_H/v_J)^\theta\). The values of \(v_H/v_J\) and \((v_H/v_J)^\theta\) are universal for a given class of M-models in a given dimension: They only depend on \(M, d\), and of course on the FP (we shall denote \(\theta_G\) and \(\theta_{CR}\) the values corresponding to \(G_{FP}\) and \(CR_{FP}\)). The glassy FP is attractive in all directions, whereas the critical one is attractive in all but one, which heads toward either \(G_{FP}\) or \(L_{FP}\). The speed at which a small perturbation from the critical manifold grows determines the critical exponent \(\nu\) of the correlation length \(\xi \sim (T - T_c)^{-\nu}\). To compute \(\nu\), we measure how two renormalized flows of the observable \(v_H/v_J\), corresponding to different original \(v_H\) and starting sufficiently near to \(CR_{FP}\) distance themselves. The exponent \(\nu\) only depends on \(M\) and \(d\) and grows for \(d \rightarrow d_1(M)\). We show values of the exponents in Table S1. Note that since \(\theta < d - 1\) even for \(G_{FP}\), the low-temperature glass phase is nontrivial. As for spin glasses, \(\theta < d - 1\) means that the system is critical (i.e., characterized by long-range critical correlations), even for \(T < T_c\).\(^4\)

### Lower Critical Dimension and Avoided Criticality

We now focus on the dependence of \(d_1(M)\) on \(M\). We find that by increasing \(M\), the lower critical dimension diminishes (i.e., glassiness is more robust for larger \(M\) values). This is also what is found within mean-field theory, and it is natural because for \(M \rightarrow \infty\), one recovers the Kac limit of mean-field models of glasses.\(^3\) Note that large-\(M\) systems could be more representative of glasses formed by interacting particles, for which the degrees of freedom are continuous (corresponding naively to \(M \rightarrow \infty\)). In Fig. 2 Insert, the values of \(d_1(M)\) are drawn as a function of \(M\) up to \(M = 7\). By performing an exponential fit, we obtain by extrapolation the value \(d_1(\infty) = 4.18 \pm 0.14\). The prediction of MK-RG is therefore that 3D disordered spin models of glasses do not display a true glass transition. This, however, does not mean that they are not glassy. Indeed, we find a static length scale \(\xi\) that grows by lowering the temperature and eventually saturates to an \(M\)-dependent value. The saturation value of \(\xi\) grows with \(M\) and diverges when \(d \uparrow d_1(M)\). The main result of this analysis is that the growth of \(\xi\) in three dimensions is due to the existence of \(CR_{FP}\) and \(G_{FP}\) FPs in higher dimensions. This is a concrete realization of the avoided transition scenario discussed previously: Since the RG flow is regular in \(d\), the existence of FPs at \(d_1\) influences the RG flow, even at lower \(d\) (the larger is \(M\), the lower is \(d_1\), and the stronger is the influence).

Our numerical analysis shows that, for temperature close to the \(T_c\) of the \(d_1\)-dimensional systems the probability distributions of \(J, H_L\), and \(H_B\) of the \((d < d_1)\)-dimensional systems first approach the ones characteristic of \(CR_{FP}\) for \(d = d_1\) and then flow toward the ones characteristic of the liquid FP. Similarly, by lowering the temperature further, the RG flow is attracted more and more toward the vestige of the \(G_{FP}\) and then eventually heads toward the liquid FP. This behavior has to stop, and indeed does so, below a crossover temperature since there is no real \(G_{FP}\). This is the mechanism that induces the growth of the length-scale \(\xi\) and, hence, assuming that energy barriers scales as the interaction, is also the one that produces glassy slow dynamics (\(\xi\) corresponds to the length \(\ell\) at which the flow starts heading toward \(L_{FP}\)). Note that the avoided transition mechanism plays an important role, even for moderate values of \(M\), for which \(d_1\) is not close to three. For example, for the finite-dimensional REM models \(M = 3\), for which \(d_1(3) \approx 6.21\), we find a static length that grows from 1 to 20 lattice spacing in three dimensions (Fig. 2). Similar results are obtained for the other \(M\)-value models. Since in experimentally relevant conditions dynamic length scales grow at best from 1 to 10 in molecular units, and static ones are believed to be even smaller (but giving rise to extremely large times via the exponential relation \(\tau \sim \eta_0 e^{\xi_0/T}\))\(^4\), 20 is not at all a small value. In conclusion, the FPs of the RG flow for \(d = d_1\) could rule the 3D glassy behavior observed in experiments and simulations, even though \(d_1\) is quite larger than three. Our MK-RG analysis suggests that the existence of a glass transition in three dimensions is a quantitative, and not a qualitative, issue: We do not find any fundamental reason to exclude the existence of an ideal glass transition in general. Indeed, it does take place if the dimension is high enough. The issue is just how low is the value of \(d_1(\infty)\), for which MK-RG only provides an approximated value (larger than three).

### Relations with Previous RG Approaches

With the aim of clarifying the state of the art and indicating new ways to proceed forward, we now compare our findings and procedure to previous works on RG analysis of the glass transition. The one of ref. 25 is the replica counterpart of ours: It consisted of performing the MK-RG directly on the replica field theory. It did find the existence of zero-temperature FPs, but those were of simple nature: a discontinuity FP for the transition and a standard one for the low-temperature phase, both with

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\(^1\) The case \(M = 1\) is an exception because of an additional (spin-flip) symmetry. In this case, there is an additional FP, placed at \((T/v_J, v_H/v_J) = (0, 0)\). \(S_G\), that is stable on the line \(v_H = 0\), but unstable on the line \(T = 0\). For \(M = 1\), if we start with \(v_H = 0\), the flow goes to \(S_G\). This result is natural because the case \(M = 1\) corresponds to a spin glass in field that is known to have a transition in zero field for \(d \geq 2\) (46-48) and that was shown by us to display a MK-RG phase diagram with four fixed points in high enough dimension (49).

\(^3\) Note that even for \(M \rightarrow \infty\), the system is non-mean-field-like on large length-scales: The limit \(d \uparrow \infty\) for the RG flow do not commute. For large, but finite, \(M\), the system is mean-field-like on large but finite length-scales only; even larger ones, the behavior is non mean-field-like and determined by the RG flow.
Inset the correlation length initially grows, but at a certain temperature it saturates to a constant value. (Inset) Lower critical dimension for the $M$-value models as a function of $M$. The curve is the exponential function used to fit the data. This fit gives an extrapolation for the lower critical dimension in the $M \to \infty$ limit to be $d_s(M = \infty) = 4.18 \pm 0.14$. The straight line reports this extrapolation.

$$\theta = d - 1.$$ Moreover, the transition was found to persist down to $d = 2$. We understand now that the saddle-point method used to solve the integral-RG equations is not fully justified. Physically, it misses the competition between the fluctuations of the interaction and of the idiosyncratic disorder crucial in our results. In ref. 23, Stevenson et al. were able to take this into account by constructing a constrained replica free-energy functional for the overlap that maps directly onto the Hamiltonian of a ferromagnetic random-field random-bond Ising model. By analyzing this model through real space RG, they found that, although in principle the disorder can destroy the transition, it does not for realistic glass models. The advantage of our approach is that we apply directly the real space RG to a microscopic model, whereas Stevenson et al. assume a particular form of the replica field theory. The advantage of their method is that it is closer to realistic systems; in particular, the bare parameters are fixed by using numerical simulations of realistic glass-formers; the drawback is that the form assumed for the replica field theory might be too restrictive. Combining their and our approaches is certainly a very interesting direction for future research. Another RG approach developed in recent years for glasses is the one based on Dyson HLs. Although a full analysis of the corresponding RG equations has not been worked out yet, the results obtained for the finite-dimensional REM do show the existence of a glass transition characterized by nontrivial critical exponents (24). It would be very interesting to go back to these studies and analyze whether the FPs are zero-temperature ones, as predicted by our work. It was recently shown, both in the case of the RFIM (51) and of spin-glasses in a field (49), that Dyson RG has the potentiality of displaying such features. The third important RG approach to glasses, which was developed by Moore and collaborators, is based on the real space MK-RG method used in this work (22, 26). They investigated the $M = 2$ and $M = 3$ $M - p$ models and found evidence of avoided criticality. Furthermore, on the basis of their previous work (26), they claimed that the critical behavior of disordered models of glasses is the same universal class of spin-glasses in a field. Our findings confirm their results, except the last one. The analysis presented in this work, which is based on a larger class of models and many different dimensions, reveals that the nature of the zero-temperature critical point and the general structure of the RG flow is different from the one surmised in refs. 22 and 26. In fact, there is no spin-glass-like zero-temperature FP as soon as $M > 1$ in any dimension (compare and contrast the RG flow of Fig. 1 to the one of spin-glasses in a field obtained in ref. 49). For small values of $M$, since the transition is strongly avoided and because of the proximity to the $M = 1$ case, which is indeed spin-glass-like, the explanation by Moore et al. could indeed be valuable, but not for larger $M$, where glass-like physics sets in. Another scenario that they put forward in ref. 57 is that it is the avoidance of the Gardner transition (11–13), which they surmise would exist in a high enough dimension, that governs the RG flow in three dimensions. This is not incompatible with our results, even though it differs from our interpretation.

**Discussion and Conclusion**

We have presented a real-space approximate RG analysis of disordered models of glasses, which are at the basis of RFOT theory. Our analysis unveils the existence and the nature of the RG FPs responsible for glassy behavior: two zero-temperature FPs, one for the transition and the other for the glass state. It suggests that, in three dimensions, the glass transition is actually avoided, but that glassiness is still driven by the RG FPs present in dimensions higher than $d_s$. For low enough temperature, the characteristic energy of the renormalized spins—and, consequently, the characteristic energy barriers—grows under renormalization as $\Delta = \ell^\theta$, where $\ell$ is the scale over which fluctuations have been integrated out by RG, until the static correlation length is reached $\xi \sim \ell$. On scales larger than $\xi$, the system is an ensemble of weakly interacting block-spins, each one characterized by the energy scale $\Delta(\xi)$. Each renormalized spin can be thought of as representative of a renormalized region of size $\ell$. A cooperative rearrangement of such a region will cost an energy of the order of $\Delta(\xi)$. The time scale for relaxation should therefore be given by the Arrhenius law applied to each subpart $\tau \approx \tau_0 e^{\Delta(\xi)/T}$. It would certainly be interesting to complement our study by using another real-space method, the Dyson Ensemble RG (58), which has the potentiality of being more accurate in high dimensions. In our study of spin-glasses in a field, corresponding to the $M = 1$ case, we did it already and found good agreement (49). Even more interesting, but certainly more challenging, would be to develop a replica RG method able to connect the mean-field RFOT results to the RG ones. In fact, although the MK-RG provides a scenario that is compatible with RFOT theory, the numerical simulations of the $M = p = 3$ model in three dimensions (52, 53) indeed show a physics more spin-glass-like.

**Note that the avoidance of a critical continuous transition leads to glassy phenomenology, as explained in ref. 29. Because of the approximate character of MK-RG, we cannot conclude whether the transition found for $d_s - d < d_s$ is a purely continuous or mixed one, as in RFOT theory. We do expect that for large $d_s$ for which mean-field results become more reliable, it should be a mixed one.

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$^4$The numerical simulations of the $M = p = 3$ model in three dimensions (52, 53) indeed show a physics more spin-glass-like.

$^5$Although the Boltzmann weight indeed increases and can be made very large from the very first RG iterations by taking $M$ large enough, the saddle-point method with a usual replica symmetry breaking structure used in ref. 25 is unable to describe the disordered nature of the nontrivial zero-temperature FPs.
key quantities, such as the configurational entropy or the overlap distribution between low-temperature glassy states, cannot be probed directly, but only inferred. For instance, the behavior of the specific heat at the transition (for $d > d_c$) is not discontinuous, which means that if the configurational entropy vanishes at $T_c$, a la Kauffman, it does so more rapidly than linearly, as is also found within the Dyson RG (24). In conclusion, given the approximate character of the MK-RG, there is some uncertainty on the true value of $d_c$. Moreover, particle models could have a lower value of $d_c$ due to an enhanced stability of the glassy phase, as argued in ref. 59. To firmly assess whether it is larger or smaller than three and to confirm and complement our results, it is crucial to develop a complete and more controlled RG analysis of the glass transition applicable both to spins and particles models. The present work provides the basis and the guidelines to face this challenge.

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