Simple nuclear C^*-algebras not isomorphic to their opposites

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We show that it is consistent with Zermelo–Fraenkel set theory with the axiom of choice (ZFC) that there is a simple nuclear nonseparable C*-algebra, which is not isomorphic to its opposite algebra. We can furthermore guarantee that this example is an inductive limit of unital copies of the Cuntz algebra $\mathcal{O}_2$ or of the canonical anticommutation relations (CAR) algebra.

C^*-algebras | Jensen's diamond | opposite algebra | Naimark's problem | Glimm dichotomy

1. Introduction

The opposite algebra of a C^*-algebra A is the C^*-algebra whose underlying Banach space structure and involution are the same as that of A, but the product of x and y is defined as yx rather than xy. It is denoted by A^{op}. In ref. 1, Connes constructed examples of factors not isomorphic to their opposites. Phillips used Connes' results in ref. 2 to construct simple separable examples, and Phillips and Viola in ref. 3 constructed a simple separable exact example. In the nuclear setting, one can construct nonsimple examples (4, 5); however, the simple nuclear case remained open both in the separable and in the nonseparable settings.

The separable case remains a difficult open problem. Approximately finite dimensional (AF) algebras are necessarily isomorphic to their opposites, due to Elliott's classification theorem, and our results show that Elliott's theorem cannot be recast as a result purely of a local approximation property. There has been major progress in the Elliott classification program recently, but the state-of-the art classification theorems all assume the Universal Coefficient Theorem (UCT). Notably, we do not know if there are Kirchberg algebras which are not isomorphic to their opposites. If such an algebra exists, then it would necessarily be a counterexample to the UCT. More generally, both the Elliott invariant and the Cuntz semigroup of any C^*-algebra A are isomorphic to that of A^{op}.

The additional axiom we add to Zermelo–Fraenkel set theory with the axiom of choice (ZFC) is Jensen's $\diamondsuit_{\aleph_1}$, discussed below in section 3, and our construction is motivated by the work of Akemann and Weaver from ref. 6, where they use $\diamondsuit_{\aleph_1}$ to construct a counterexample to the Naimark problem. Our main theorem is:

Theorem 1.1. *Assume $\diamondsuit_{\aleph_1}$ holds. Then there exists a nuclear, simple, unital C^*-algebra A not isomorphic to its opposite algebra.*

In fact, we obtain the following strengthening.

Theorem 1.2. *Assume $\diamondsuit_{\aleph_1}$ holds and $1 \leq n \leq \aleph_0$ is given. Then there exists a C^*-algebra A with the following properties.*

(1) *A is nuclear, simple, unital and of density character $\aleph_1$.*
(2) *A is not isomorphic to its opposite algebra.*
(3) *A has exactly n unitarily nonequivalent irreducible representations.*
(4) *All automorphisms of A are inner.*

In addition, one can ensure that one of the following holds.

(5) *A is an inductive limit of subalgebras isomorphic to the Cuntz algebra $\mathcal{O}_2$.*
(6) *A is an inductive limit of subalgebras isomorphic to full matrix algebras of the form $M_{2^n}(\mathbb{C})$.*

By Glimm's theorem (see the remark in the second paragraph from the end of p. 586 of ref. 7), every separable and simple C^*-algebra with nonequivalent irreducible representations has $2^{\aleph_0}$ nonequivalent irreducible representations. Item (*3*) above shows that the failure of this dichotomy for nonseparable C^*-algebras is relatively consistent with ZFC.

The observation that the proof of ref. 6 gives a nuclear counterexample to Naimark's problem is due to N. C. Phillips. We don't know whether a simple, nuclear C^*-algebra not isomorphic to its opposite can be constructed in ZFC, and whether a counterexample to Naimark's problem can be constructed in ZFC. Another problem raised by our proof of Theorem 1.2 is whether a counterexample to Naimark's problem can have an outer automorphism.

We use the following notation throughout. We count 0 as a natural number. If $\mathcal{Y} = \langle a_j : j \in \mathbb{N}\rangle$ is a sequence of elements in some set, we denote by $b^\frown\mathcal{Y}$ the sequence whose first element is b and whose $j+1$st element is a_j.

2. Extending States

This section contains technical lemmas which will be used in the induction step of our construction. We first give a modification of a lemma of Kishimoto, Lemma 2.2, and a toy version, Lemma 2.1.

Lemma 2.1. *Let A be a non-type I, separable, simple, unital C^*-algebra. Let C and D be nonzero hereditary subalgebras of A, and let $\varepsilon > 0$. Let $n \geq 1$ and let $u_0, u_1, \ldots u_n$ be some elements in A^+. Then there exist positive elements $c \in C$ and $d \in D$ of norm 1 such that $\|cu_k d\| < \varepsilon$ for $k = 0, 1, \ldots, n$.*

Significance

The Hilbert space ℓ^2 is the (usually infinite-dimensional) modification of our standard three-dimensional space. C*-algebras are suitably closed algebras of linear operators on ℓ^2. The algebras of complex $n \times n$ matrices are the simplest examples of C*-algebras. The opposite of a C*-algebra is the algebra in which the direction of the multiplication is reversed. Although every matrix algebra is isomorphic to its opposite, we construct an inductive limit of matrix algebras not isomorphic to its opposite. This algebra is an example of a simple amenable C*-algebra not isomorphic to its opposite. Our examples can have exactly n inequivalent irreducible representations for any n, showing that Glimm's dichotomy can fail for simple nonseparable C*-algebras.

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Proof: We denote $A_\infty := l^\infty(\mathbb{N}, A)/C_0(\mathbb{N}, A)$, and we identify A with the subalgebra given by constant sequences. As A is not a continuous trace algebra, by ref. 8, theorem 2.4, the central sequence algebra $A_\infty \cap A'$ is nontrivial. Let $x \in A_\infty \cap A'$ be a self-adjoint element whose spectrum has more than one point. Because A is simple, the C^*-algebra generated by x and A inside of A_∞ is isomorphic to $C(\sigma(x)) \otimes A$, and therefore, if $y \in C^*(x)$ and $a \in A$, then $\|ya\| = \|y\|\|a\|$. Because $\sigma(x)$ has more than one point, we may pick $y, z \in C^*(x)_+$ with norm 1 such that $yz = 0$. Pick $(y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}} \in l^\infty(\mathbb{N}, A)_+$ which lift y and z, respectively. Fix elements $c_0 \in C_+$ and $d_0 \in D_+$ of norm 1. Then $\lim_{n\to\infty}\|c_0^{1/2} y_n c_0^{1/2}\| = \lim_{n\to\infty}\|d_0^{1/2} z_n d_0^{1/2}\| = 1$, and $\lim_{n\to\infty} \|c_0^{1/2} y_n c_0^{1/2} \cdot u_k \cdot d_0^{1/2} z_n d_0^{1/2}\| = \lim_{n\to\infty}\|c_0 y_n z_n u_k d_0\| = 0$. For all sufficiently large n, the elements $c = \frac{1}{\|c_0^{1/2} y_n c_0^{1/2}\|} \cdot c_0^{1/2} y_n c_0^{1/2}$ and $d = \frac{1}{\|d_0^{1/2} z_n d_0^{1/2}\|} \cdot d_0^{1/2} z_n d_0^{1/2}$ satisfy the requirements. □

Lemma 2.2. *Suppose A is a non-type I, separable, simple, unital C^*-algebra and suppose α is an antiautomorphism of A or an outer automorphism of A. Then for any nonzero hereditary C^*-subalgebra B of A and every unitary $u \in A$ we have*

$$\inf\{\|bu\alpha(b)\| : b \in B_+, \|b\| = 1\} = 0.$$

Proof: Because an automorphism of a simple C^*-algebra is outer if and only if its Connes spectrum is distinct from $\{1\}$, the case in which α is an outer automorphism is a special case of ref. 9, lemma 1.1.

Suppose α is an antiautomorphism and let $\alpha' := Adu \circ \alpha$. By ref. 10, theorem 1, we have $\inf\{\|b\alpha'(b)\| : b \in B_+, \|b\| = 1\} = 0$. But $\|b\alpha'(b)\| = \|bu\alpha(b)u^*\| = \|bu\alpha(b)\|$ and the conclusion follows. □

Lemma 2.3. *Suppose A is a separable, simple, unital C^*-algebra. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are disjoint countable sets of unitarily nonequivalent pure states of A and suppose E is an equivalence relation on $\mathcal{Y}$. Then there exists a separable simple unital C^*-algebra C with the following properties.*

(1) *A is a unital subalgebra of C.*
(2) *Every $\psi \in \mathcal{Y}$ has a unique extension $\tilde\psi$ to a pure state of C.*
(3) *If ψ_0 and ψ_1 are in $\mathcal{Y}$ then $\psi_0 \mathrel{E} \psi_1$ if and only if $\tilde\psi_0$ and $\tilde\psi_1$ are unitarily equivalent pure states of C.*
(4) *Every $\psi \in \mathcal{X}$ has more than one extension to a pure state of C.*

In addition, if $A \cong \mathcal{O}_2$ then one can arrange $C \cong \mathcal{O}_2$.

Proof: We shall construct an automorphism β of A of infinite order such that the crossed product $C := A \rtimes_\beta \mathbb{Z}$ is as required. By ref. 6, theorem 2, a pure state φ of A has a unique extension to a pure state of C if and only if φ is nonequivalent to $\varphi \circ \beta^n$ for all $n \neq 0$. Because A is non-type I and separable, by Glimm's theorem it has $2^{\aleph_0}$ nonequivalent pure states. We can therefore extend $\mathcal{Y}$ to ensure that every E-equivalence class is infinite and that there are infinitely many equivalence classes. We can similarly assume $\mathcal{X}$ is infinite. Let π_j^k, for $j \in \mathbb{Z}$, be an enumeration of Gelfand–Naimark–Segal (GNS) representations corresponding to states in the k-th E-equivalence class. Let σ_j, for $j \in \mathbb{N}$, be an enumeration of GNS representations corresponding to states in $\mathcal{X}$. All of these representations correspond to pure states and are therefore irreducible. By the extension of ref. 11 proved in ref. 6 (pp. 7523–7524), there exists an automorphism β of A such that

(5) π_j^k is equivalent to $\pi_l^m \circ \beta$ if and only if $k = m$ and $j = l + 1$.
(6) σ_j is equivalent to $\sigma_j \circ \beta$ for all j.

By ref. 9, theorem 3.1, the crossed product $C := A \rtimes_\beta \mathbb{Z}$ is simple. By ref. 6, theorem 2, it satisfies (*1*), (*2*), and (*4*).

To prove (*3*), fix ψ_0 and ψ_1 in $\mathcal{Y}$. If $\psi_0 \mathrel{E} \psi_1$ then (*6*) implies that the unique pure state extensions of ψ_0 and ψ_1 to C are equivalent. Now suppose ψ_0 and ψ_1 are not E-related. Then ψ_0 and $\psi_1 \circ \beta^n$ are inequivalent for all $n \in \mathbb{Z}$. To get a contradiction, suppose that the unique pure state extensions of ψ_0 and ψ_1 to C are equivalent and let v be a unitary in C such that $\psi_0 = \psi_1 \circ Adv$. Let u be the canonical unitary implementing β. Approximate v up to $1/2$ by a finite linear combination $\sum_{n=-k}^{k} c_n u^n$, where $c_n \in A$. Choose decreasing sequences a_j, b_j, for $j \in \mathbb{N}$, of positive elements of norm 1 such that the a_j excise ψ_0 and the b_j excise ψ_1 (ref. 12, proposition 2.2). Note that $\beta^n(b_j)$ excises $\psi_1 \circ \beta^{-n}$ for all n. By ref. 6, lemma 1, for all $x \in A$ we have $\|a_j x \beta^n(b_j)\| \to 0$ as $j \to \infty$. Thus, for j large enough, we have $\|a_j c_n \beta^n(b_j)\| < 1/(4k+2)$ for all $-k \le n \le k$. Then $\|a_j v b_j v^*\| = \|a_j v b_j\| < 1$. On the other hand, the Cauchy–Schwarz inequality implies $\psi_0(a_j v b_j v^*) = \psi_0(v b_j v^*) = \psi_1(b_j) = 1$; contradiction.

Finally, if $A \cong \mathcal{O}_2$, then $C = A \rtimes_\beta \mathbb{Z} \cong \mathcal{O}_2$. One way to establish this isomorphism is to note that by (*5*) above, no nonzero power of β is inner; therefore by ref. 13, theorem 1 the automorphism β has the Rokhlin property, hence by ref. 14, theorem 4.4 we have $C \cong C \otimes \mathcal{O}_2$, so by ref. 15, theorem 3.8 we have $C \cong \mathcal{O}_2$. □

The following is a strengthening of ref. 9, theorem 2.1.

Lemma 2.4. *Suppose A is a non-type I, separable, simple, unital C^*-algebra, and suppose α is an antiautomorphism, or an outer automorphism. Then there exists a family $\mathcal{W}$ of $2^{\aleph_0}$ pure states of A such that φ is not unitarily equivalent to $\varphi \circ \alpha$ for every $\varphi \in \mathcal{W}$.*

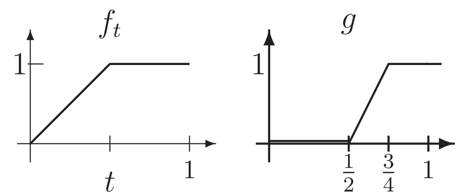
Proof: The proofs in the case when α is an outer automorphism and when α is an antiautomorphism differ very little and will be presented simultaneously.

Let u_n, for $n \in \mathbb{N}$, be an enumeration of a dense set of unitaries of A. By $\{0,1\}^{<\mathbb{N}}$ we denote the set of all finite sequences of $\{0,1\}$ ordered by the end-extension, denoted $s \sqsubset t$. The empty sequence $\langle\rangle$ is the minimal element of $\{0,1\}^{<\mathbb{N}}$, its immediate successors are 0 and 1, and the immediate successors of $s \in \{0,1\}^{<\mathbb{N}}$ are $s^\frown 0$ and $s^\frown 1$. The length of $s \in \{0,1\}^{<\mathbb{N}}$ is denoted $|s|$.

Given $\delta \in (0, 1/2)$, we claim that there exist $a(s)$ and $e(s)$ in A_+, for $s \in \{0,1\}^{<\mathbb{N}}$ and $j = 0, 1$ for $s \in \{0,1\}^{<\mathbb{N}}$, with the following properties:

(1) $\|a(s)\| = \|e(s)\| = 1$.
(2) $a(s)e(s^\frown j) = e(s^\frown j)$.
(3) $e(s)a(s) = a(s)$.
(4) $\|e(s^\frown 0)e(s^\frown 1)\| < \delta$.
(5) $\|e(s^\frown 0)u_k e(s^\frown 1)\| < \delta$ for all $k \le |s|$.
(6) $\|a_s u_{|s|} \alpha(a_s)\| < \delta$.

The family $\{e(s), a(s)\}_{s\in\{0,1\}^{<\mathbb{N}}}$ will be constructed by recursion. Define $f_t, g : [0,1] \to [0,1]$ for $t \in (0,1)$ as follows.



Notice that $f_{1/2} \cdot g = g$, and $\|f_t - id\| = 1 - t$. Fix $\varepsilon \in (0, 1/2)$ such that whenever x, y are positive contractions in some C^*-algebra and z is any contraction such that $\|xzy\| < \varepsilon$ then $\|f_{1/2}(x) z f_{1/2}(y)\| < \delta$ and $\|g(x)zg(y)\| < \delta$. (That such ε exists can be shown using polynomial approximations for $f_{1/2}$ and for g.)

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Let $a(\langle\rangle)=1$. Suppose $a(s)$ was chosen. By Lemma 2.1 applied to $n=|s|+1$ and the unitaries u_k for $k\leq n$, there exist $h_0,h_1\in B(s)_+$ such that $\|h_0\|=\|h_1\|=1$ and $\|h_0u_kh_1\|<\varepsilon$. for all $k\leq|s|$. Let

$$e(s^\frown j):=f_{1/2}(h_j).$$

By Lemma 2.2 there exists $a(s^\frown j)\in\overline{g(h_j)Ag(h_j)}_+$ that satisfies $\|a(s^\frown j)\|=1$ and $\|a(j)u_{|s|}\alpha(a(j))\|<\delta$. We may assume without loss of generality, that there exists a nonzero positive element $b(s^\frown j)$ with $a(s^\frown j)b(s^\frown j)=b(s^\frown j)$ (by replacing $a(s^\frown j)$ by $f_t(a(s^\frown j))$ for t sufficiently close to 1 if need be).

The family $\{e(s),a(s)\}_{s\in\{0,1\}}^{<\mathbb{N}}$ satisfying (*1*)–(*6*) can now be constructed by using a standard bookkeeping device. Fix an enumeration s_j, for $j\in\mathbb{N}$, for $\{0,1\}^{<\mathbb{N}}$ such that $s_j\sqsubset s_k$ implies $j<k$ (e.g., let $\{s\in\{0,1\}^{<\mathbb{N}}:|s|=n\}$ be enumerated as s_j, for $2^{n-1}\leq j<2^n$). By using the above, one can recursively find $e(s_j)$ and $a(s_j)$ for $j\in\mathbb{N}$ in the hereditary subalgebra on which all of the elements of the form $e(s)$ and $a(s)$, where $s\sqsubset s_j$, act as the identity.

Denote the set of all infinite sequences of $\{0,1\}$ by $\{0,1\}^{\mathbb{N}}$. For $h\in\{0,1\}^{\mathbb{N}}$ let $h\restriction n$ denote the initial segment of h of length n, for $n\in\mathbb{N}$. For $h\in\{0,1\}^{\mathbb{N}}$ we have $h\restriction n\in\{0,1\}^{<\mathbb{N}}$ and

$$\mathcal{F}(h):=\{a(h\restriction n):n\in\mathbb{N}\}$$

is a sequence of elements of A_+ of norm 1 such that

$$a(h\restriction n)a(h\restriction(n+1))=a(h\restriction(n+1))$$

for all n. Hence

$$\{\zeta\in\mathcal{S}(A):\zeta(a)=1\text{ for all }a\in\mathcal{F}(h)\}$$

is a face of $\mathcal{S}(A)$. Let ζ_h be an extreme point of this face; then ζ_h is a pure state of A satisfying $\zeta_h(a(h\restriction n))=1$ for all n. By (*3*) we have $\zeta_h(e(h\restriction n))=1$ for all n and thus, by the Cauchy–Schwarz inequality, we have $\zeta_h(e(h\restriction n)b)=\zeta_h(b)$ for all b and for all n.

We claim that the states ζ_h and $\zeta_{h'}$ are not unitarily equivalent if $h\neq h'$. Suppose otherwise. Then for some $j\in\mathbb{N}$ we have $\|\zeta_h-\zeta_{h'}\circ Adu_j\|<1/2$. Fix $n\geq j$ large enough to have $h\restriction n\neq h'\restriction n$. By (*5*) we have $\|e(h\restriction n)Adu_j(e(h'\restriction n))\|<\delta<1/2$, but $|\zeta_h(e(h\restriction n)Adu_j(e(h'\restriction n))|=|\zeta_h(Adu_j(e(h'\restriction n))|>1/2$, a contradiction.

By the same argument and (*6*), ζ_h is not equivalent to $\zeta_h\circ\alpha$ for every $h\in\{0,1\}^{\mathbb{N}}$. We should note that whether α is an automorphism or an antiautomorphism, it preserves the order structure of A and it is an affine homeomorpism of $\mathcal{S}(A)$ onto itself. Therefore, $\zeta_h\circ\alpha$ is a pure state of A. □

The next few technical lemmas will be used to construct a uniformly hyperfinite (UHF) example.

Definition 2.5: Suppose A is a separable UHF algebra. A family of pure states $\langle\varphi_n:n\in\mathbb{N}\rangle$ of A will be called *separated product states* if there exist $\langle k(n):n\in\mathbb{N}\rangle$, a map Φ, subalgebras A_n, and projections $\langle p_{n,j}:n\in\mathbb{N},j<n\rangle$ and $\langle q_n:n\in\mathbb{N}\rangle$ with the following properties.

(1) $k(n)\geq1$, for $n\in\mathbb{N}$.
(2) $\Phi:A\to\otimes_nM_{k(n)}(\mathbb{C})$ is an isomorphism.
(3) $A_n:=\otimes_{j<n}M_{k(j)}(\mathbb{C})$.
(4) $p_{n,j}$, for $0\leq j<n$, are orthogonal rank 1 projections in $M_{k(n)}(\mathbb{C})$, for all n,
(5) $q_m\in A_m$ is a rank-1 projection, and
(6) φ_m is the product state of $A_n\otimes\otimes_{j=m+1}^{\infty}M_{k(j)}(\mathbb{C})$ uniquely determined by the requirement that for all $l\geq1$ we have

$$\varphi_m(q_m\otimes p_{m+1,m}\otimes p_{m+2,m}\otimes\cdots\otimes p_{m+l,m})=1.$$

Lemma 2.6. *Suppose A is a UHF algebra and π_n for $n\in\mathbb{N}$, are irreducible representations of A. Then the following are equivalent.*

(1) *$\langle\pi_n:n\in\mathbb{N}\rangle$ are pairwise nonequivalent irreducible representations of A,*
(2) *There are separated product states φ_n, for $n\in\mathbb{N}$, such that π_n is the GNS representation corresponding to φ_n for all n.*

Proof: Suppose φ_j, for $j\in\mathbb{N}$, are separated product states of a UHF algebra. For all $j\neq l$ and $n\in\mathbb{N}$ there exists a projection $p\in A_n'\cap A$ such that $\varphi_j(p)=0$ and $\varphi_l(p)=1$, and therefore ref. 16, theorem 3.4 implies that φ_l is not unitarily equivalent to φ_j for $j\neq l$.

Now suppose π_j, for $j\in\mathbb{N}$, are as in (*2*). Let ψ_j be a pure state such that π_j is the GNS representation corresponding to ψ_j for $j\in\mathbb{N}$. Let φ_j, for $j\in\mathbb{N}$, be a sequence of separated pure states of A. By (*1*) these pure states are nonequivalent and by the extension of ref. 11 proved in ref. 6 (pp. 7523–7524) (or, because A is UHF, by ref. 17, theorem 7.5) there exists an automorphism β of A such that $\varphi_j=\psi_j\circ\beta$ for all $j\in\mathbb{N}$, as required. □

We need the following variant of Lemma 2.3 for the CAR algebra, M_{2^∞}.

Lemma 2.7. *Suppose $A\cong M_{2^\infty}$. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are disjoint countable sets of unitarily nonequivalent pure states of A and E is an equivalence relation on $\mathcal{Y}$. Then there exists a separable simple unital C^*-algebra C with the following properties.*

(1) *$C\cong M_{2^\infty}$.*
(2) *A is a unital subalgebra of C.*
(3) *Every $\psi\in\mathcal{Y}$ has a unique extension $\tilde\psi$ to a pure state of C,*
(4) *If ψ_0 and ψ_1 are in $\mathcal{Y}$ then $\psi_0\,E\,\psi_1$ if and only if $\tilde\psi_0$ and $\tilde\psi_1$ are unitarily equivalent pure states of C.*
(5) *Every $\psi\in\mathcal{X}$ has more than one extension to a pure state of C.*

Proof: We shall first provide a proof in case when E is the identity relation on $\mathcal{Y}$. By Lemma 2.6 we may identify A with $\otimes_nM_{k(n)}(\mathbb{C})$ witnessing that the pure states in $\mathcal{X}\cup\mathcal{Y}$ are separated. Because $A\cong M_{2^\infty}$, for every n there exists $l(n)\in\mathbb{N}$ such that $k(n)=2^{l(n)}$. We may assume that $k(n)>2n$ for all n. In $M_{k(n)}(\mathbb{C})$ we have n orthogonal rank 1 projections $p_{n,j}$, for $j\leq n$, each corresponding to a unique state in $\mathcal{X}\cup\mathcal{Y}$. Let $\mathcal{P}$ be a maximal family of orthogonal rank 1 projections in $M_{k(n)}$ including $\{p_{n,j}:j\leq n\}$. Because $k(n)>2n$, we can find a permutation σ of $\mathcal{P}$ such that

(6) $\sigma(p_{n,j})=p_{n,j}$ if and only if $p_{n,j}$ corresponds to a pure state in $\mathcal{X}$,
(7) $\sigma(p_{n,j})\neq p_{n,k}$ if $p_{n,j}$ and $p_{n,k}$ correspond to distinct pure states in $\mathcal{Y}$, and
(8) $\sigma^2=id_{\mathcal{P}}$.

Let $u_n\in M_{k(n)}(\mathbb{C})$ be an order 2 unitary such that $Adu_n(q)=\sigma(q)$ for all $q\in\mathcal{P}$ and such that $Tr(u_n)=0$. (One can construct such a unitary by first considering a permutation matrix corresponding to σ, and noting that the number of 1's on the diagonal must be even; we then define u_n to be a matrix obtained by starting out with this permutation matrix and replacing half of the 1's on the diagonal by -1's.) Note that the automorphism $\beta:=\otimes_nAdu_n$ also satisfies $\beta^2=id_A$.

Set A_n as in Definition 2.5. Each A_n is β-invariant, and we have $A\rtimes_\beta\mathbb{Z}/2\mathbb{Z}=\overline{\bigcup_nA_n\rtimes_{\beta|_{A_n}}\mathbb{Z}/2\mathbb{Z}}$. Note that $A_n\rtimes_{\beta|_{A_n}}\mathbb{Z}/2\mathbb{Z}\cong A_n\oplus A_n$, and the inclusion

$$A_n\rtimes_{\beta|_{A_n}}\mathbb{Z}/2\mathbb{Z}\to A_{n+1}\rtimes_{\beta|_{A_{n+1}}}\mathbb{Z}/2\mathbb{Z}\cong(A_n\rtimes_{\beta|_{A_n}}\mathbb{Z}/2\mathbb{Z})\otimes M_{k(n)}$$

is given by a direct sum of $k(n)/2$ copies of the identity map, and $k(n)/2$ copies of the map $a\oplus b\mapsto b\oplus a$. Thus, by considering the Bratteli diagram of this AF system, we see that $A\rtimes_\beta\mathbb{Z}/2\mathbb{Z}\cong M_{2^\infty}$.

By ref. 6, theorem 2, a pure state φ of A has a unique extension to a pure state of C if and only if φ and $\varphi\circ\beta$ are not unitarily equivalent. By the choice of u_n and β, a pure state $\varphi \in \mathcal{X} \cup \mathcal{Y}$ has a unique extension to a pure state of C if and only if $\varphi \in \mathcal{X}$. If φ and ψ are distinct and belong to $\mathcal{Y}$, then by (7) for every finite-dimensional subalgebra B of C, there exists a projection $p \in B' \cap C$ (one can choose it of the form $q + \sigma(q)$ for q which corresponds to ψ) such that $\tilde{\varphi}(p) = 0$ and $\tilde{\psi}(p) = 1$. Therefore, ref. 16, theorem 3.4 implies that $\tilde{\varphi}$ is not unitarily equivalent to $\tilde{\psi}$.

We now consider the case when E is a nontrivial equivalence relation on $\mathcal{Y}$. Enumerate the i-th E-equivalence class as $\langle \zeta_j^i : j < n \rangle$, for some $1 \le n \le \aleph_0$. In the above construction there is sufficient room for us to choose the symmetry σ so the resulting automorphism β satisfies $\zeta_0^i \circ \beta = \zeta_1^i$ for all i. The resulting crossed product, A_1, is isomorphic to M_{2^∞}, every $\zeta_j^i \in \mathcal{Y}$ has a unique extension $\tilde{\zeta}_j^i$ to a pure state of A_1, and $\tilde{\zeta}_j^i$ is equivalent to $\tilde{\zeta}_k^l$ if and only if $i = l$ and $\max(j, k) \le 1$. We can now apply this construction to A_1, with $\mathcal{X} := \emptyset$, $\mathcal{Y} := \{\tilde{\zeta}_j^i : j \ge 1\}$ and E defined by $\tilde{\zeta}_j^i E \tilde{\zeta}_k^l$ if and only if $i = l$ and $\min(j, k) \ge 1$ and obtain a crossed product A_2. After at most $\aleph_0$ steps, all E-equivalence classes will be taken care of. The inductive limit C of A_n is, by the classification of AF algebras, isomorphic to M_{2^∞} and it has all of the required properties. □

The following lemma serves as the inductive step in our construction.

Lemma 2.8. *Suppose A is a non-type I, separable, simple, unital C^*-algebra and let $\mathcal{Y}$ be a countable set of pure states of A. Let ζ be a pure state of A which is not unitarily equivalent to any of the states in $\mathcal{Y}$. Suppose α is an antiautomorphism, or an outer automorphism, of A. Then there exist a separable simple unital C^*-algebra C and a pure state ψ of C such that:*

(1) *A is a unital C^*-subalgebra of C.*
(2) *Each $\varphi \in \mathcal{Y}$ has a unique extension to a pure state of C, and those unique extensions are pairwise unitarily inequivalent.*
(3) *ζ has a unique extension to a pure state in C which is unitarily equivalent to the extension of some pure state from $\mathcal{Y}$.*
(4) *ψ is the unique extension of some pure state in $\mathcal{Y}$.*
(5) *α cannot be extended to an antiautomorphism or an automorphism of C.*
(6) *If a C^*-algebra D has C as a subalgebra and ψ has a unique state extension to D then α cannot be extended to an antiautomorphism or an automorphism of D.*

In addition, if $A \cong \mathcal{O}_2$ then we can arrange $C \cong \mathcal{O}_2$, and if $A \cong M_{2^\infty}$ then we can arrange $C \cong M_{2^\infty}$.

Proof: Again, the proofs in the case in which α is an outer automorphism and when α is an antiautomorphism differ very little and will be presented simultaneously. We note in passing that our assumptions imply that A is nonabelian, hence an automorphism of A cannot be extended to an antiautomorphism of C and vice versa; however, this fact is unimportant for the proof.

Because the given set $\mathcal{Y}$ of pure states is countable, by Lemma 2.4, we can choose a pure state ψ_0 such that for any $\varphi \in \mathcal{Y} \cup \{\zeta\}$, neither ψ_0 nor $\psi_1 := \psi_0 \circ \alpha$ is unitarily equivalent to φ. Let $\mathcal{Y}' := \mathcal{Y} \cup \{\zeta, \psi_0\}$, and define an equivalence relation E on $\mathcal{Y}'$ such that $\zeta \, E \, \varphi$ and $\psi_0 \, E \, \varphi$ for some $\varphi \in \mathcal{Y}$, and all other elements of $\mathcal{Y}'$ are equivalent via E only to themselves. We then apply Lemma 2.3 or 2.7 to $\mathcal{X} = \{\psi_1\}$ and $\mathcal{Y}'$ to obtain a C^*-algebra C (with $C \cong A$ if A is M_{2^∞} or $\mathcal{O}_2$) such that ψ_0, ζ and all $\varphi \in \mathcal{Y}$ have unique pure state extensions to C, ψ_1 has multiple state extensions to C, and the unique extensions of ψ_0 and ζ are equivalent to the unique extension of some $\varphi \in \mathcal{Y}$; the latter state is ψ as in (6).

Suppose D is a C^*-algebra that has C as a C^*-subalgebra, and assume that α extends to $\tilde{\alpha}$ which is an automorphism or an antiautomorphism of D. If ψ has a unique state extension $\tilde{\psi}$ to D, then $\tilde{\psi} \circ \tilde{\alpha}$ is the unique extension of ψ_1 to D. As ψ_1 has multiple state extensions to C this is a contradiction, and therefore (6) holds. □

3. Diamond and the Construction

A subset $\mathcal{C}$ of $\aleph_1$ is called *closed and unbounded* (*club*) if for every $\eta < \aleph_1$ there exists $\xi \in \mathcal{C}$ such that $\xi > \eta$, and for every countable $X \subseteq \mathcal{C}$ we have $\sup(X) \in \mathcal{C}$ (ref. 18, section III.6). A subset $\mathcal{S}$ of $\aleph_1$ is *stationary* if it intersects every club nontrivially. Because the intersection of two clubs (and even countably many clubs) is a club, the intersection of a stationary set with a club is again stationary. We shall use von Neumann's definition of an ordinal as the set of all smaller ordinals.

Jensen's $\diamondsuit_{\aleph_1}$ asserts that there exists a family of sets S_ξ, for $\xi < \aleph_1$, such that

(1) $S_\xi \subseteq \xi$ for all $\xi < \aleph_1$, and
(2) for every $X \subseteq \aleph_1$ the set $\{\xi : X \cap \xi = S_\xi\}$ is stationary.

This combinatorial principle is true in Gödel's constructible universe L (see, e.g., ref. 18, section III.7.13) and is therefore relatively consistent with ZFC. A much easier fact is that it implies the Continuum Hypothesis (see, e.g., ref. 18, section III.7.2).

Although $\diamondsuit_{\aleph_1}$ captures subsets of $\aleph_1$, it is well-known among logicians that $\diamondsuit_{\aleph_1}$ implies its self-strengthening, which captures countable subsets of any algebraic structure in countable signature of cardinality $\aleph_1$. This observation extends to metric structures. Because we could not find a reference for this fact in the literature, we work out the details in case of C^*-algebras equipped with some additional structure.

Suppose A is a C^*-algebra with a given sequence of states $\mathcal{Y} = \langle \varphi_j : j \in \mathbb{N} \rangle$ and a linear isometry $\alpha : A \to A$. (We are interested in the case when α is an automorphism or an antiautomorphism.) Suppose we are given a dense subset of A, $\mathbb{A} := \{a_\xi : \xi < \theta\}$, indexed by an ordinal θ. In addition, suppose that $\mathbb{A}$ is closed under $+, \cdot, {}^*$, α, and multiplication by the complex rationals, $\mathbb{Q} + i\mathbb{Q}$. Consider the following subsets of θ^k, for $1 \le k \le 3$ and of $\theta \times \mathbb{Q}$:

(1) $\mathbb{A}(+) := \{(\xi, \eta, \mu) \in \theta^3 : a_\xi + a_\eta = a_\mu\}$,
(2) $\mathbb{A}(\cdot) := \{(\xi, \eta, \mu) \in \theta^3 : a_\xi a_\eta = a_\mu\}$,
(3) $\mathbb{A}({}^*) := \{(\xi, \eta) \in \theta^2 : a_\xi^* = a_\eta\}$,
(4) $\mathbb{A}(\|\cdot\|) := \{(\xi, r) \in \theta \times \mathbb{Q}_+ : \|a_\xi\| \ge r\}$,
(5) $\mathbb{A}(\mathbb{C}) := \{(\xi, \eta) \in \theta^2 : a_\xi = i a_\eta\}$,
(6) $\mathbb{A}(\varphi_j) := \{(\xi, r) \in \theta \times \mathbb{Q} : \varphi_j(a_\xi^* a_\xi) \ge r\}$, for $j \in \mathbb{N}$,
(7) $\mathbb{A}(\alpha) := \{(\xi, \eta) \in \theta^2 : \alpha(a_\xi) = a_\eta\}$.

This countable family of sets uniquely determines a countable normed algebra over $\mathbb{Q} + i\mathbb{Q}$ whose completion is isomorphic to A. It also uniquely determines both α and the sequence $\mathcal{Y}$. We say that the structure $(A, \mathbb{A}, \alpha, \varphi : \varphi \in \mathcal{Y})$ is *coded* by $\mathfrak{X} := \langle \mathbb{A}(\bullet) : \bullet \in \{+, \cdot, {}^*, \|\cdot\|, \mathbb{C}, \alpha, \varphi : \varphi \in \mathcal{Y}\} \rangle$ and construe the latter as a subset of

$$\mathbb{X}(\theta) := \theta^3 \sqcup \theta^3 \sqcup \theta^2 \sqcup \theta \times \mathbb{Q} \sqcup \theta^2 \sqcup \theta \times \mathbb{Q} \times \mathcal{Y} \sqcup \theta^2.$$

Clearly, $\mathbb{X}(\theta)$ and θ have the same cardinality for any infinite θ.

A nested transfinite sequence A_ξ, for $\xi < \aleph_1$, of C^*-algebras is said to be *continuous* if for every limit ordinal $\eta < \aleph_1$ we have $A_\eta = \overline{\bigcup_{\xi < \eta} A_\xi}$.

Lemma 3.1. *$\diamondsuit_{\aleph_1}$ implies that there exists a family $\{T_\xi\}_{\xi < \aleph_1}$ such that:*

(1) *$T_\xi \subseteq \mathbb{X}(\xi)$ for all $\xi < \aleph_1$,*
(2) *for every continuous nested family $\{A_\xi\}_{\xi < \aleph_1}$ of separable C^*-algebras, for any enumeration $\{a_\xi | \xi < \aleph_1\}$ of $A = \varinjlim A_\xi$, for any countable set $\mathcal{Y}$ of pure states of A and for any linear isometry α of A onto A, the set of all $\theta < \aleph_1$ such that*

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(a) $\varphi \restriction A_\theta$ *is pure for all* $\varphi \in \mathcal{Y}$,
(b) $\alpha(A_\theta) = A_\theta$, *and*
(c) T_θ *codes the structure* $(A_\theta, \{a_\xi : \xi < \theta\}, \alpha \restriction A_\theta, \varphi \restriction A_\theta : \varphi \in \mathcal{Y})$ *is stationary.*

Proof: Fix a bijection $f : \aleph_1 \to \mathbb{X}(\aleph_1)$. Writing $f[X] := \{f(x) : x \in X\}$, define $g : \aleph_1 \to \aleph_1$ by $g(\xi) := \min\{\eta : f[\xi] \subseteq \mathbb{X}(\eta), f^{-1}[\mathbb{X}(\xi)] \subseteq \eta\}$. (Because every countable subset of $\aleph_1$ is bounded, g is well-defined.) The set of fixed points of g, $\mathcal{C} := \{\theta < \aleph_1 : g[\theta] = \theta\}$, is a club (ref. 18, lemma III.6.13) and $\mathcal{C} \subseteq \{\theta < \aleph_1 : f[\theta] = \mathbb{X}(\theta)\}$. Let $\{S_\xi\}_{\xi < \aleph_1}$ be a family of sets as in the definition of $\diamondsuit_{\aleph_1}$. We claim that $T_\xi := f[S_\xi]$, for $\xi \in \mathcal{C}$, and $T_\xi := \emptyset$, for $\xi \notin \mathcal{C}$, are as required. (Many of the T_ξ don't code anything resembling a C^*-algebra, but this fact is of no concern for us.)

Suppose $A = \lim_{\to} A_\xi$, $\mathcal{Y}$, α, and $\{a_\xi : \xi < \aleph_1\}$ are as in (*2*). Set $\mathbb{A}_\theta := \{a_\xi : \xi < \theta\}$. Note that the set

$$\mathcal{C}_0 := \{\theta < \aleph_1 : \mathbb{A}_\theta \text{ is a dense } \mathbb{Q} + i\mathbb{Q} \text{ subalgebra of } A_\theta\}$$

is a club. Because the intersection of countably many clubs is a club, ref. 6, lemma 4, implies that

$$\mathcal{C}_1 := \{\theta \in \mathcal{C}_0 : \varphi_j \restriction A_\theta \text{ is pure for all } j \in \mathbb{N} \text{ and } \alpha[A_\theta] = A_\theta\}$$

is also a club. Let $\mathfrak{X} \subseteq \mathbb{X}(\aleph_1)$ be the code of $(A, \mathbb{A}, \alpha, \varphi : \varphi \in \mathcal{Y})$ and with f used to define T_ξ, let $X := f^{-1}(\mathfrak{X})$. By $\diamondsuit_{\aleph_1}$, the set $\{\theta : X \cap \theta = S_\theta\}$ is stationary, and therefore so is its intersection with $\mathcal{C}_1$. But $\{\theta : X \cap \theta = S_\theta\} \cap \mathcal{C}_1$ is precisely the set of ordinals θ which satisfy (*2*), as required. □

Proof of Theorem 1.2: We construct a continuous nested sequence $\{A_\eta : \eta < \aleph_1\}$ of simple, separable unital and nuclear C^*-algebras and inequivalent pure states φ^j_η, for $j < n$, of A_η, such that φ^j_η and φ^j_ξ agree on A_ξ if $\xi < \eta$. Because $\diamondsuit_{\aleph_1}$ implies the Continuum Hypothesis, each A_η as well as $\bigcup_{\eta<\aleph_1} A_\eta$ will be of cardinality $\aleph_1$. We shall choose an enumeration $A_\eta = \{b^\xi_\eta : \xi < \aleph_1\}$ for every η and a countable dense subset $\mathbb{A}_\eta = \{a^\eta_\xi : \xi < \eta\}$ of A_η for every limit ordinal η such that

(1) $\mathbb{A}_\eta$ is closed under $+, \cdot, ^*$, and multiplication by the complex rationals, $\mathbb{Q} + i\mathbb{Q}$,
(2) $a^\zeta_\xi = a^\eta_\xi$ if $\xi < \zeta < \eta$ and ζ and η are limit ordinals,
(3) $\{b^\xi_\zeta : \max\{\xi, \zeta\} < \eta\} \subseteq \mathbb{A}_\eta$.

We begin with $A_0 = \mathcal{O}_2$ or $A_0 = M_{2^\infty}$ and any fixed (finite or infinite) sequence $\langle \varphi^j_0 : j < n\rangle$ of inequivalent pure states of A_0.

If θ is a limit ordinal, then we let $A_\theta := \lim_{\xi<\theta} A_\xi$ and let φ^j_θ be the unique state extending all φ^j_ξ for $\xi < \theta$ for $j < n$; this state is necessarily pure. If in addition θ is a limit of limit ordinals, then $\mathbb{A}_\theta$ is already uniquely determined and conditions (*2*) and (*3*) for $\zeta < \eta < \theta$ imply the corresponding conditions for $\eta < \theta$. If θ is a limit ordinal, but not a limit of limit ordinals, then the supremum of limit ordinals $< \theta$ is the largest limit ordinal below θ; we denote it by η. Then, the set $\{\xi : \eta \le \xi < \theta\}$ is infinite. Because A_θ is separable and the set on the left-hand side of (*3*) is countable, $\mathbb{A}_\theta$ can be defined so that it satisfies the requirements.

Now suppose θ is a successor ordinal, say, $\theta = \xi + 1$. To proceed from A_ξ to $A_{\xi+1}$, we first check whether there exists an outer automorphism or an antiautomorphism α of A_ξ, pure state ψ of A_ξ, and (if n is finite) an extension of $\langle \varphi^j_\xi : j < n\rangle$ to an infinite sequence $\mathcal{W}$ such that $(A_\xi, \mathbb{A}_\xi, \psi^\frown\mathcal{W}, \alpha)$ is coded by T_ξ. If so, let $A_{\xi+1}$ be the C^*-algebra C given by Lemma 2.8 in which the unique extension of ψ is unitarily equivalent to a unique extension of some φ^j_ξ. Let $\varphi^j_{\xi+1}$ be the unique extension of φ^j_ξ, for $j < n$. If T_ξ does not code such $(A_\xi, \mathbb{A}_\xi, \psi^\frown\mathcal{W}, \alpha)$, let $A_{\xi+1} := A_\xi$. This process describes the construction.

Let A be the inductive limit of this nested sequence. It is nuclear, simple, and unital, being the inductive limit of simple nuclear C^*-algebras with unital connecting maps. Using (*2*) we can write $a_\xi := a^\zeta_\xi$ for ζ being any limit ordinal greater than ξ. Because $A = \bigcup_\xi A_\xi$ by (*3*) we have $A = \{a_\xi : \xi < \aleph_1\}$.

The sequence of pure state extensions φ^j_θ defines n inequivalent pure states φ^j, for $j < n$, of A. These states have the property that φ^j is a unique extension of φ^j_θ to A, for every $\theta < \aleph_1$. If n is finite, let $\mathcal{W}$ be any infinite sequence of pure states of A extending $\langle \varphi^j : j < n\rangle$.

Suppose $A_0 \cong \mathcal{O}_2$ and $A_\xi \cong \mathcal{O}_2$ for all $\xi < \theta$. If $\theta = \xi + 1$, then $A_\theta \cong \mathcal{O}_2$ because it was obtained by using Lemma 2.8. If θ is a limit ordinal, then ref. 19, corollary 5.1.5, implies $A_\theta \cong \mathcal{O}_2$. Therefore, by induction $A_\xi \cong \mathcal{O}_2$ for all $\xi < \aleph_1$. Likewise, if $A_\xi \cong M_{2^\infty}$ for all $\xi < \theta$ then $A_\theta \cong M_{2^\infty}$ by the classification of AF algebras (noting that the inclusion maps all induce an isomorphism on the K_0 groups). Because A has density character $\aleph_1$, it is an inductive limit of full matrix algebras by ref. 20, theorem 1.3 (1).

Suppose that A has an antiautomorphism or an outer automorphism α and let φ be any pure state of A. Then, there exists $\theta < \aleph_1$ such that $(A_\theta, \mathbb{A}_\theta, \varphi^\frown\mathcal{W} \restriction A_\theta, \alpha \restriction A_\theta)$ was coded by T_θ at stage θ. Hence $A_{\theta+1}$ was produced by using Lemma 2.8, and there exists $j < n$ such that $\alpha \restriction A_\theta$ cannot be extended to an antiautomorphism or an outer automorphism of any C^*-algebra which contains $A_{\theta+1}$ and to which $\varphi^j_{\theta+1}$ has a unique state extension. By construction, this state has a unique extension to A_η for all $\eta \ge \theta + 1$ and therefore it has a unique extension to A. But α clearly extends $\alpha \restriction A_\theta$; contradiction.

We already know that A has at least n inequivalent pure states. Let ψ be any pure state of A. With $\alpha = id_A$, there exists $\theta < \aleph_1$ such that $(A_\theta, \mathbb{A}_\theta, \alpha \restriction A_\theta, \varphi \restriction A_\theta)$ was coded by T_θ at stage θ. Hence, $A_{\theta+1}$ was produced by using Lemma 2.8 and $\varphi \restriction A_\theta$ has a unique extension to $A_{\theta+1}$ equivalent to $\varphi^j_{\theta+1}$ for some $j < n$. Because φ^j is the unique extension of the latter to a state of A, we conclude that ψ is equivalent to φ^j. Because ψ was arbitrary, we conclude that every pure state of A is equivalent to some φ^j, for $j < n$, and therefore A has exactly n inequivalent pure states. □

Remark 3.2: The AF algebra we constructed is not isomorphic to an (uncountable) infinite tensor power of copies of M_2 (or M_n). To see that, notice that an infinite tensor product of matrix algebras is the complexification of a real C^*-algebra (namely, the corresponding infinite tensor product of $M_2(\mathbb{R})$). A complexification of a real C^*-algebra is always isomorphic to its opposite (any real C^*-algebra is isomorphic to its opposite via the * map, which is $\mathbb{R}$-linear, which one can then complexify).

Remark 3.3: Our construction is C^*-algebraic in nature. It does, however, raise the analogous question for von-Neumann algebras: is there a hyperfinite factor (with nonseparable predual) which is not isomorphic to its opposite? More concretely, our AF example has unique trace. Let M be the weak closure of its image under the GNS representation. Is M isomorphic to its opposite? A peculiar hyperfinite II_1 factor with no nontrivial central sequences was constructed by using the Continuum Hypothesis in ref. 21.

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