Trisections of 4-manifolds

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The study of n-dimensional manifolds has seen great advances in the last half century. In dimensions greater than four, surgery theory has reduced classification to homotopy theory except when the fundamental group is nontrivial, where serious algebraic issues remain. In dimension 3, the proof by Perelman of Thurston's Geometrization Conjecture (1) allows an algorithmic classification of 3-manifolds. The work of Freedman (2) classifies topological 4-manifolds if the fundamental group is not too large. Also, gauge theory in the hands of Donaldson (3) has provided invariants leading to proofs that some topological 4-manifolds have no smooth structure, that many compact 4-manifolds have countably many smooth structures, and that many noncompact 4-manifolds, in particular 4D Euclidean space R^4, have uncountably many.

However, the gauge theory invariants run into trouble with small 4-manifolds, such as those with the same homology groups as the 4D sphere, S^4. In particular, the smooth 4D Poincaré Conjecture, the last remaining case of that hallowed conjecture, is still open. (In higher dimensions, the smooth Poincaré Conjecture is sometimes true in the following sense. In dimensions 3, 5, 6, 12, and 61, a homotopy sphere is diffeomorphic to the standard one, and in all other known cases, there are increasingly many exotic smooth structures on the topological sphere; however, it is possible that there may be more high-dimensional cases with no exotic spheres.) The gauge theory invariants are very good at distinguishing smooth 4-manifolds that are homotopy equivalent but do not help at showing that they are diffeomorphic. What is missing is the equivalent of the higher-dimensional s-cobordism theorem, a key to the successes in higher dimensions.

The s-cobordism theorem states that, if M_0^m and M_1^m are the two boundary components of an $m+1$-dimensional manifold W and if W is the simple homotopy type of $M_0 \times [0,1]$, then W is homeomorphic to $M_0 \times [0,1]$, and thus, M_0 is homeomorphic to M_1. This requires that $m > 4$, or if $m = 4$, then the fundamental group is good. If the manifolds are smooth, then homeomorphism can be replaced by diffeomorphism if $m > 4$. We have assumed that the manifolds are closed, connected, and oriented. Note that, if $\pi_1(M_0) = 0$, then simply homotopy equivalent is the same as homotopy equivalent. The point to this theorem is that algebraic topological invariants are enough to produce homeomorphisms, a geometric conclusion.

In dimension 3, stronger results now hold due to Perelman (e.g., simple homotopy equivalence implies diffeomorphism).

However, in dimension 4, we only have powerful invariants that distinguish different smooth structures on 4-manifolds but nothing like the s-cobordism theorem, which can show that two smooth structures are the same. Conjecturally, trisections of 4-manifolds may lead to progress.

Trisections of 4-manifolds are analogous to Heegaard splittings of 3-manifolds. The latter describe a closed, oriented, connected 3-manifold Y as the union of two handlebodies along their common boundaries, a surface Σ_g of genus g, where a handlebody is a 0-handle (just a 3-ball) and g 1-handles. A Heegaard diagram is Σ_g together with two sets, α and β, of g disjointly embedded circles, where each set cuts Σ into a $2g$ punctured sphere and each circle bounds a 2-ball embedded in its handlebody, thus cutting the handlebodies into 3-balls (Fig. 1).

Heegaard diagrams naturally arise from a Morse function $f: M \to [0,3]$, where the critical points of index 0 and 3 are taken by f to 0 and 3, the Heegaard surface Σ is the preimage of the regular value 1.5, the critical points of index 1 have values in $(0,1)$, and the critical points of index 2 have values in $(2,3)$. A critical point of index 1 has an ascending disk, and its boundary is a circle in Σ belonging to α (similarly with critical points of index 2 and their descending disks).

A trisection τ of X^4 (a smooth, connected, closed, oriented 4-manifold) is then a splitting of X into three parts, $X_1 \cup X_2 \cup X_3$, each of which is a 4D handlebody (a connected sum of k copies of $S^1 \times B^3$ s), where

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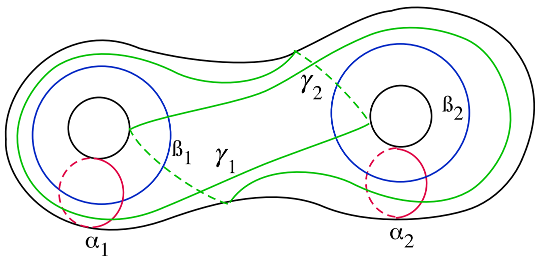

Fig. 1. This shows a surface of genus two which bounds the evident handlebody, as well as the three sets of curves defining a trisection of $S^2 \times S^2$.

$X_i \cap X_j$ is a 3D handlebody and $X_1 \cap X_2 \cap X_3 = \Sigma_g$. Note that $(X_i \cap X_j) \cup (X_j \cap X_k) = \partial X_j$ is a connected sum of k copies of $S^1 \times S^2$, where Σ_g is the Heegaard surface for a genus g Heegaard splitting of ∂X_j, $i \neq j \neq k$. (Trisections arose from ref. 4 and appear in ref. 5.)

A trisection diagram arises from a Morse 2-function from X to the disk: $f: X \to B^2$. Over any generic radius of B^2 is a 3-manifold, namely the connected sum of g copies of $S^1 \times B^2$, and this is obtained by adding g 2-handles to Σ and a 3-handle. Three cases of this are evident, $X_i \cap X_j$, $i \neq j \in \{1, 2, 3\}$, and we name the attaching circles of the 2-handles in $X_1 \cap X_2$ by $\alpha = \alpha_1, \alpha_2. \ldots \alpha_g$ (similarly for the other two cases named by β and γ).

Now, we can draw a diagram on B^2 indicating the critical values of the Morse 2-function f. These are fold curves (Fig. 2 shows a diagram for the complex projective plane CP^2), where in local coordinates $f(t, x, y, z) = (t, -x^2 - y^2 \pm z^2)$; t parameterizes the fold curve, and the coefficient of z^2 is negative for the outside definite fold curve (indicating an arc of minima) and is positive for the other g indefinite fold curves. The cusps correspond to points that, in the right coordinates, are just births of a pair of canceling critical points of index 1 and index 2. The ascending circle of the index 1 critical point intersects the descending circle of the index 2 critical point in exactly one point, as when an α curve intersects a β curve in one point.

Monodromy

The first paper in this feature (6) exploits the diagram of a trisecction of X by constructing a monodromy that measures the minimal number of handle slides that occur when following, say, the α handlebody as it moves counterclockwise around B^2. We know that, say, the α and β curves can be put in standard position, where each α_i has a counterpart β_i that is parallel or they intersect in exactly one point and β_i intersects no other α. Similarly, the γ and β can be put in standard position, but some of the β must slide over other β to accomplish this (similarly with the α and γ). The monodromy counts the total minimal number of such slides required to bring the α handlebody back to itself, and if the monodromy is zero or one, then X must be one of four elementary 4-manifolds. Unfortunately, the monodromy is very hard to compute.

4-Manifolds with Boundary

Trisections for 4-manifolds with boundary were defined, and the existence and uniqueness theorems for the closed case were easily extended to the case with boundaries in ref. 5. The simplest case occurs when ∂X is a surface bundle over the circle with fiber Σ. Then, the obvious map to S^1 extends over X to B^2. However, to get the general case, use is made of the fact that ∂X can always be described as an open book.

This case is greatly extended in the second paper in this feature (7). The key idea is that the boundary data should be those of an open book decomposition on the 3D boundary Y (i.e., a decomposition into a link $L \subset Y$, the binding, and a surface bundle of $Y - L$ over S^1, with fibers called the pages, where the boundary of each page is the binding L). The simplest example of an open book is the 3-sphere, where the binding L is a great circle with complement that is a circle crossed with an open disk; the disks are then the pages.

Given two trisected 4-manifolds with the same boundary, if the induced open book decompositions agree, then the trisections glue to give a trisection of the resulting closed 4-manifold (8).

In the second paper by Castro et al. (7), it is shown how to turn an explicit handle diagram of a 4-manifold with boundary, together with an explicit description of the bounding open book decomposition, into a trisection diagram for the 4-manifold inducing the given open book. Furthermore, this can be done in the case of multiple boundary components, which should be useful for constructions in which one glues along some but not all boundary components.

Classical Invariants

If X^4 is described as a trisection τ, then one would want to know how to calculate the classical invariants of X using the data of τ. The paper by Feller et al. (9) shows how to determine the homology groups as well as the intersection form and signature of a 4-manifold that is described as a trisection. Explicit computations are given to show the techniques for some simple trisection diagrams.

Group Trisections

Applying the fundamental group functor to a Heegaard splitting of M^3, one gets a square of epimorphisms: one for each map of the Heegaard surface of genus g to each of the 3D handlebodies [giving $\pi_1(\Sigma_g) = S_g \to F_g = \pi_1(H_i)$, $i = 1, 2$.] and then, two epimorphisms onto $\pi_1(M)$.

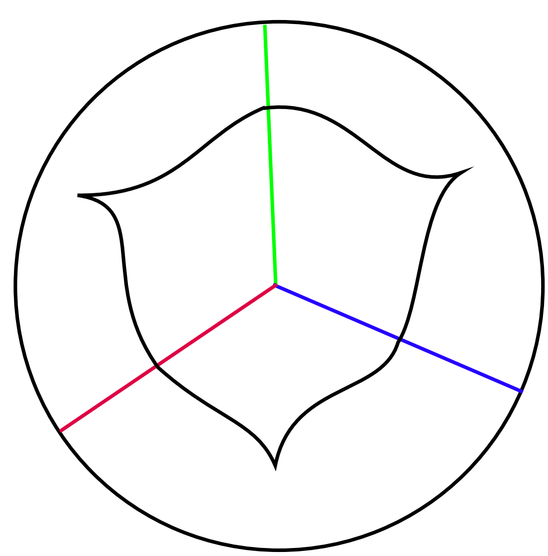

Fig. 2. This diagram shows the outside definite fold curve and the indefinite fold curve with three cusps, representing the complex projective plane.

$$
\begin{tikzcd}
 & F_g \arrow[r] \arrow[dr] & Z_k \arrow[dr] & \\
S_g \arrow[ur] \arrow[r] \arrow[dr] & F_g \arrow[ur] \arrow[dr] & Z_k \arrow[r] & G \\
 & F_g \arrow[r] \arrow[ur] & Z_k \arrow[ur] &
\end{tikzcd}
$$

Fig. 3. A trisection of a finitely presented group G represented by a cube of epimorphisms.

In a similar way, a trisection τ of X^4 gives a cube of epimorphisms by using the three maps of Σ to the 3D handlebodies $X_i \cap X_j, i \neq j = 1, 2, 3$, the maps taking $X_i \cup X_j, i = 1, 2, 3$ to the 4D handlebodies X_i, $i = 1, 2, 3$, and finally, an epimorphism to $\pi_1(X^4)$. This cube is called a trisection of a group G (Fig. 3), and it turns out that any finitely presented group G has trisections and that they correspond in a one-to-one fashion with smooth, closed, orientable 4-manifolds (10).

In his paper, Klug (11) shows that this correspondence between trisected 4-manifolds and trisections of groups is in fact functorial.

Bridge Trisections of Knotted Surfaces in 4-Manifolds

An important facet of understanding smooth 4-manifolds involves studying embedded surfaces, which can be knotted in this dimension. For example, the minimal genus among surfaces representing a fixed second homology class carries significant information about the smooth topology of the ambient 4-manifold. In another vein, various surgery operations—wherein a neighborhood of a knotted surface is excised from the ambient 4-manifold only to be replaced in a controlled way—can be used to relate diverse 4-manifolds and construct interesting new ones.

In a trisected 4-manifold, a natural question is whether any embedded surface can be isotoped to meet the components of the trisection in some standard way. The main result of this paper (12) is that this can always be achieved. Given a trisected 4-manifold and an embedded surface therein, the surface can be isotoped to lie in bridge trisected position (i.e., to intersect the 3D handlebodies of the trisection in trivial collections of arcs and to intersect the 4D handlebodies in trivial collections of disks). (Here, trivial means boundary parallel.) This builds on previous work of Meier and Zupan (13). What follows is an interesting interplay between the complexity of the bridge trisected surface and the original trisection of the 4-manifold.

Consequently, a trisected surface can be diagrammatically encoded on the central surface of the trisection. This gives a diagrammatic theory for studying knotted surfaces in 4-manifolds, including, for example, the fact that any knotted sphere in a 4-manifold can be represented by a doubly pointed trisection diagram (Fig. 4).

Dehn Surgery

One way to describe a given smooth n-manifold X is to build X from a set of fundamental pieces, called k-handles, along with a set of instructions about how to put these pieces together. In dimension 4, attaching a 2-handle to a compact manifold along its boundary Y is completely determined by a framed knot K in Y, and the 3D boundary of the resulting manifold can be described by removing a solid torus neighborhood V of K and reattaching it by gluing a meridian curve of V to the framing curve of K. This process is called Dehn surgery, and it is intimately tied to the study of 4D handle decompositions.

This paper (14) is devoted to the study of n-component links in S^3 with a Dehn surgery yielding the 3-manifold $\#^n(S^1 \times S^2)$, called an R-link. An R-link L can be interpreted as the gluing instructions for building a closed 4-manifold X_L that is homotopy equivalent to S^4. The Smooth 4-Dimensional Poincaré Conjecture asserts that every such 4-manifold is diffeomorphic to S^4. The Generalized Property R Conjecture (GPRC), Kirby Problem 1.82, states that every R-link can be converted into an unlink by a sequence of handleslides. In this paper, the authors describe a three-step program to disprove the GPRC. First, they convert the statement of the GPRC into a statement about trisections of smooth 4-manifolds following previous work by Meier et al. (15). Second, they describe trisection diagrams for the most prominent potential counterexamples to the GPRC. Third, the last step, which remains incomplete, is to prove that a family of these trisections does not admit a certain type of destabilization operation. They offer the rectangle condition as a way to certify that a trisection cannot be destabilized as partial progress in this direction.

Lefschetz Fibrations

Complex surfaces have been studied for centuries in algebraic geometry and have provided fascinating examples for 4D topologists (e.g., the complex surfaces of degree d in CP^3, the best known of which is the quartic or K3 surface). A well-studied family of complex surfaces is the Lefschetz fibrations, complex analytic maps $X \to CP^1$ that are bundle maps except for Lefschetz singularities, which in local coordinates, are given by $(z, w) \to zw$. Broken Lefschetz fibrations are more general, because they allow the fibers (real surfaces) to change genus when crossing fold curves in CP^1.

Simplified trisections are trisections that arise from Morse 2-functions to the disk with embedded singular images. Remarkably, the existence and stable uniqueness results of Gay–Kirby for trisections on arbitrary 4-manifolds also hold for this subclass (16). In their article in this feature, Baykur and Saeki (17) study (simplified) trisections from the vantage point of singularity theory and explain how to derive a simplified trisection from a (broken) Lefschetz fibration via topological modifications and vice versa. This yields several new constructions of interesting families of trisections, such as an infinite family of genus 3 trisections on rational homology 4-spheres.

Computations

There are several common ways of describing a smooth 4-manifold X. Among the most frequent perhaps are handlebody

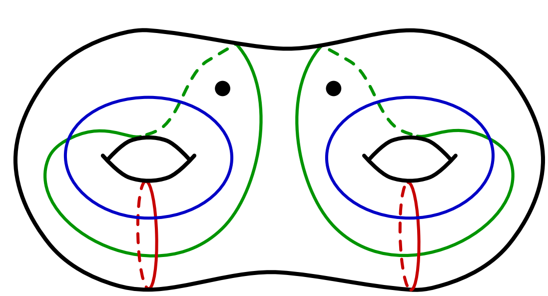

Fig. 4. A trisection of a knotted 2-sphere embedded in the 4-sphere.

decompositions arising from Morse functions and as a union of 4D simplices glued together along their 3D faces. This is always possible and is unique up to subdivision and simplicial isomorphism. These are sometimes called piecewise linear (PL) structures on X, and a special role is played by combinatorial triangulations, which reflect the manifold structure.

The paper of Bell et al. (18) starts with a PL structure and provides a practical algorithm to determine a trisection and a trisection diagram from a triangulation. The heart of the construction is a PL map from the triangulated 4-manifold to the 2-simplex, such that the dual cubical structure of the 2-simplex pulls back to a trisection. The algorithm has been implemented and used to establish minimal trisections of the standard simply connected 4-manifolds in ref. 19.

Generalized Trisections

One can imagine trying to generalize trisections while keeping a 2D central fiber, like Σ_g, and allowing a higher-dimensional base, say B^n, so that we have $f: X^{n+2} \to B^n$. Alternatively, we could keep a 2D base and allow an n-dimensional central fiber.

Rubinstein and Tillmann (20) in the last paper take a different route, starting with a suitably triangulated n-manifold N and building a map to B^k, where $n = 2k, 2k+1$. Here, B^k is viewed as a k-simplex and has a natural dual cell decomposition into $k+1$ k-cubes. The inverse image of each of these cubes is required to be an n-dimensional handlebody, and there are additional requirements on the intersections of these handlebodies. In the cases $n = 3$ and $n = 4$, this yields the classical decompositions into Heegaard splittings and trisections, respectively. In even dimensions, a key example is the moment map defined on complex projective space.

To construct a suitable triangulation, the key idea is to use a coloring scheme for the vertices of the triangulation so that each n-simplex has either $2k+1$ or $2k+2$ vertices, which have a $(k+1)$ coloring, so that there are one or two vertices of the same color. There is then a natural simplicial map to the k-simplex given by mapping vertices of the same color to the same image. Such coloring schemes are inspired by the work in ref. 21. The pull back of the cubical structure of the k-simplex is then a multisection of the n-manifold.

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