Statistical dynamical model to predict extreme events and anomalous features in shallow water waves with abrupt depth change

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Understanding and predicting extreme events and their anomalous statistics in complex nonlinear systems are a grand challenge in climate, material, and neuroscience as well as for engineering design. Recent laboratory experiments in weakly turbulent shallow water reveal a remarkable transition from Gaussian to anomalous behavior as surface waves cross an abrupt depth change (ADC). Downstream of the ADC, probability density functions of surface displacement exhibit strong positive skewness accompanied by an elevated level of extreme events. Here, we develop a statistical dynamical model to explain and quantitatively predict the above anomalous statistical behavior as experimental control parameters are varied. The first step is to use incoming and outgoing truncated Korteweg–de Vries (TKdV) equations matched in time at the ADC. The TKdV equation is a Hamiltonian system, which induces incoming and outgoing statistical Gibbs invariant measures. The statistical matching of the known nearly Gaussian incoming Gibbs state at the ADC completely determines the predicted anomalous outgoing Gibbs state, which can be calculated by a simple Markov chain Monte Carlo algorithm verified by direct numerical simulations, and successfully captures key features of the experiment. There is even an analytic formula for the anomalous outgoing skewness. The strategy here should be useful for predicting extreme anomalous statistical behavior in other dispersive media in different settings (21, 22).

1. Experiments Showing Anomalous Wave Statistics Induced by an ADC

Controlled laboratory experiments were carried out in ref. 20 to examine the statistical behavior of surface waves crossing an ADC. In these experiments, nearly unidirectional waves are generated by a paddle wheel and propagate through a long, narrow wave tank. Midway through, the waves encounter a step in the bottom topography and thus, abruptly transition to a shallower bottom topography and thus, abruptly transition to a shallower depth. The paddle wheel is forced with a pseudorandom signal intended to mimic a Gaussian random sea upstream of the ADC. In particular, the paddle angle is specified as

$$\theta(t) = \theta_0 + \Delta \theta \sum_{n=1}^{N} a_n \cos(\omega_n t + \delta_n), \quad E \sim (\Delta \theta)^2 \sum_{n} a_n^2 \omega_n^2,$$

outside the ADC completely determines the predicted anomalous outgoing Gibbs state, which can be calculated by a simple Markov chain Monte Carlo algorithm verified by direct numerical simulations, and successfully captures key features of the experiment. There is even an analytic formula for the anomalous outgoing skewness. The strategy here should be useful for predicting extreme anomalous statistical behavior in other dispersive media.

Significance

Understanding and predicting extreme events and their anomalous statistics in complex nonlinear systems are a grand challenge in applied sciences as well as for engineering design. Recent controlled laboratory experiments in weakly turbulent shallow water with abrupt depth change exhibit a remarkable transition from nearly Gaussian statistics to extreme anomalous statistics with large positive skewness of the surface height. We develop a statistical dynamical model to explain and quantitatively predict the anomalous statistical behavior. Incoming and outgoing waves are modeled by the truncated Korteweg–de Vries equations statistically matched at the depth change. The statistical matching of the known nearly Gaussian incoming Gibbs state completely determines the predicted anomalous outgoing Gibbs state and successfully captures key features of the experiment.


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where the weights $a_\omega$ are Gaussian in spectral space with peak frequency $\omega_0$ and the phases $\delta_\omega$ are uniformly distributed random variables. The peak frequency gives rise to a characteristic wavelength $\lambda_c$, which can be estimated from the dispersion relation. The energy $E$ injected into the system is determined by the angle amplitude $\Delta \theta$, which is the main control parameter varied in ref. 20. Optical measurements of the free surface reveal a number of surprising statistical features.

- Distinct statistics are found between the incoming and outgoing waves disturbances: incoming waves display near-Gaussian statistics, while outgoing waves skew strongly toward positive displacement.
- The degree of non-Gaussianity present in the outgoing waves depends on the injected energy $E$: larger energies generate stronger skewness in the surface displacement probability density functions (PDFs) and more extreme events.
- Compared with the incoming wave train, the power spectrum of the outgoing wave field decays more slowly, which indicates that the anomalous behavior is associated with an elevated level of high frequencies.

2. Surface Wave Turbulence Modeled by TKdV Equation with Depth Dependence

The Korteweg–de Vries (KdV) equation is a one-dimensional, deterministic model capable of describing surface wave turbulence. More specifically, KdV is leading-order approximation for surface waves governed by a balance of nonlinear and dispersive effects valid in an appropriate far-field limit (23). Moreover, KdV has been adapted to describe waves propagating over variable depth (23). Here, we consider the variable-depth KdV equation truncated at wavenumber $\Lambda$ with $J = 2\Lambda + 1$ grid points to generate weakly turbulent dynamics. The surface displacement is described by the state variable $u^\pm_\Lambda (x,t)$, with superscript + for the incoming waves and superscript − for the outgoing waves. The Galerkin truncated variable $u^\pm_\Lambda = \sum_{k=\Lambda}^{\pm \Lambda} \tilde{u}_k(t) e^{2\pi i k x}$ is normalized with zero mean $\tilde{u}_0 = 0$ and unit energy $2\pi \sum_{k=\pm 1} |\tilde{u}_k|^2 = 1$, which are conserved quantities. Here, $u^\pm_\Lambda \equiv \mathcal{P}_\Lambda u$ denotes the subspace projection. The evolution of $u^\pm_\Lambda$ is governed by the truncated KdV equation with depth change $D_\pm$:

$$\frac{\partial u^\pm_\Lambda}{\partial t} + \frac{D^\pm - 3/2}{2} E^{1/2}_0 \frac{\partial}{\partial x} \mathcal{P}_\Lambda \left( u^\pm_\Lambda \right)^2 + D^{1/2}_\pm L_0^{-3/2} \frac{\partial^3 u^\pm_\Lambda}{\partial x^3} = 0,$$

[1]

Eq. 1 is nondimensionalized on the periodic domain $x \in [-\pi, \pi]$. The depth is assumed to be unit $D_\pm = 1$ before the ADC and $D_\pm < 1$ after the ADC. The conserved Hamiltonian can be decomposed as

$$\mathcal{H}_\Lambda^\pm = D_\pm^{-3/2} E_0^{1/2} L_0^{-3/2} H_3(u^\pm_\Lambda) - D_\pm^{1/2} L_0^{-3} H_2(u^\pm_\Lambda),$$

where we refer to $H_3$ as the cubic term and $H_2$ as the quadratic term. We introduce parameters $(E_0, \Lambda)$ based on the following assumptions.

- The wavenumber truncation $\Lambda$ is fixed at a moderate value for generating weakly turbulent dynamics.
- The state variable $u^\pm_\Lambda$ is normalized with zero mean and unit energy, $M(\langle u^\pm_\Lambda \rangle) = 0$, $\mathcal{E}(\langle u^\pm_\Lambda \rangle) = 1$, which are conserved during evolution. Meanwhile, $\mathcal{E}_0$ characterizes the total energy injected into the system based on the driving amplitude $\Delta \theta$.
- The length scale of the system $L_0$ is chosen so that the resolved scale $\Delta x = 2\pi / L_0 / J$ is comparable with the characteristic wave length $\lambda_c$ from the experiments.

Some intuition for how Eq. 1 produces different dynamics on either side of the ADC can be gained by considering the relative contributions of $H_3$ and $H_2$ in the Hamiltonian $\mathcal{H}_\Lambda$. The depth change, $D_\pm < 1$, increases the weight of $H_3$ and decreases that of $H_2$, thereby promoting the effects of nonlinearity over dispersion and creating conditions favorable for extreme events. Since $\partial H_2 / \partial x$ is the slope of the wave height, $\mathcal{H}_2(u)$ measures the wave slope energy.

A deterministic matching condition is applied to the surface displacement $u^\pm_\Lambda$ to link the incoming and outgoing wave trains. Assuming that the ADC is set at $t = T_{ADC}$, the matching condition is given by

$$u^\pm_\Lambda (x,t) |_{t = T_{ADC}^-} = u^\pm_\Lambda (x,t) |_{t = T_{ADC}^+}.$$

Eq. 1 is not designed to capture the short scale changes in rapid time. Rather, since we are interested in modeling statistics before and after the ADC, we will examine the long-time dynamics of large-scale structures.

Interpreting Experimental Parameters in the Dynamical Model. The model parameters $(E_0, \Lambda, \lambda_c)$ in [1] can be directly linked to the basic scales from the physical problem. The important parameters that characterize the experiments (20) include $\epsilon = \gamma \lambda_c / \pi$, the wave amplitude $a$ to (upstream) water depth $H_0$ ratio; $\delta = \lambda_c \pi$, the water depth to wavelength $\lambda_c$ ratio; and $D_\pm = a / \pi$, the depth ratio with upstream value $d = H_0$ and downstream value $d < H_0$. By comparing the characteristic physical scales, the normalized TKdV Eq. 1 can be linked to these experimental parameters via

$$L_0 \sim 6 \left( M \epsilon^2 \delta^{-1} \right), \quad E_0 \sim 27 \gamma^2 \left( M \epsilon^2 \delta^{-1} \right),$$

[2]

where $M$ defines the computational domain size $M \lambda_c$, as $M$ multiple of the characteristic wavelength $\lambda_c$, and $\gamma = \frac{U}{c}$, with $U$ a scaling factor for the state variable $u^\pm_\Lambda$, normalizes the total energy of the system to 1.

Consider a spatial discretization with $J = 2\Lambda + 1$ grid points so that the smallest resolved scale is comparable with the characteristic wavelength $\Delta x = 2\pi M \lambda_c / J \sim 5$, $J = 32$.

In the practical numerical simulations, we select $M = 5$ and let $\gamma$ vary in the range $[0.5, 1]$. Using reference values from the experiments (20), we find $\epsilon \in [0.0024, 0.0244]$, $\delta \sim 0.22$, and $D_\pm$ changes from 1 to 0.24 at the ADC. These values yield the following ranges for the model parameters: $L_0 \in [2, 6]$ and $E_0 \in [50, 200]$. These are the values that we will test in the direct numerical simulations. Details about the derivation from scale analysis are in SI Appendix, Section A.

3. Equilibrium Statistical Mechanics for Generating the Stationary Invariant Measure

Since the TKdV equation satisfies the Liouville property, the equilibrium invariant measure can be described by a statistical formalism (24–26) based on a Gibbs measure with the conserved energy $\mathcal{E}_0$ and Hamiltonian $\mathcal{H}_\Lambda$. The equilibrium invariant measure is dictated by the conservation laws in the TKdV equation. In the case of fixed energy $\mathcal{E}_0$, this is the mixed Gibbs measure with microcanonical energy and canonical Hamiltonian ensembles (24)

$$G^\pm_\Lambda (u^\pm_\Lambda; E_0) = C^\pm_\Lambda \exp \left(-\theta^\pm \mathcal{H}(u^\pm_\Lambda) - \mathcal{E}(u^\pm_\Lambda) + E_0 \right),$$

[3]

where $\theta$ represents the “inverse temperature.” The distinct statistics in the upstream and downstream waves can be
controlled with $\theta$. As shown below, we find that negative temperature, $\theta^2 < 0$, is the appropriate regime to predict the experiments. In the incoming wave field, $\theta$ is chosen so that $G_{\theta}$ has nearly Gaussian statistics. Using the above invariant measures $[3]$, the expectation of any functional $F(u)$ can be computed as

$$\langle F \rangle_{G_{\theta}} \equiv \int F(u) G_{\theta}(u) du.$$ 

The value of $\theta$ in the invariant measure is specified from $\langle H_\Lambda \rangle_{G_{\theta}}$ (24, 26).

In addition to producing equilibrium PDFs of $u_\Lambda$, the invariant measure can be used to predict the equilibrium energy spectrum without the direct simulation of TKdV. Direct simulation, however, is required to recover transient statistics of $u_\Lambda$ and time autocorrelations.

**Statistical Matching Condition of the Invariant Measures Before and After the ADC.** The Gibbs measures $G_{\theta}$ are defined based on the different inverse temperatures $\theta^\pm$ on the two sides of the solutions

$$\mu^-\left( u_\Lambda ; D^-, \right), \quad u_\Lambda \big| t = T_{ADC}^- = u_0, \quad t < T_{ADC};$$

$$\mu^+\left( u_\Lambda ; D^+, \right), \quad u_\Lambda \big| t = T_{ADC}^+ = u_0, \quad t > T_{ADC},$$

where $u_0$ represents the deterministic matching condition between the incoming and outgoing waves. The two distributions, $\mu^-, \mu^+$, should be matched at $T_{ADC}$, giving

$$\mu^-\left( u_\Lambda \right) = \mu^+\left( u_\Lambda \right).$$

In matching the flow statistics before and after the ADC, we first use the conservation of the deterministic Hamiltonian $H_\Lambda^+$ after the depth change. Then, assuming ergodicity (24, 25), the statistical expectation for the Hamiltonian $\langle H_\Lambda \rangle$ is conserved in time after the depth change at $t = T_{ADC}$ and should remain at this value as the system approaches equilibrium as $t \to \infty$. The final statistical matching condition to get the outgoing flow statistics with parameter $\theta^+$ can be found by

$$\langle H_\Lambda \rangle_{G_{\theta}^+} = \langle H_\Lambda \rangle_{G_{\theta}^-}, \quad [4]$$

with the outgoing flow Hamiltonian $H_\Lambda^+$ and the Gibbs measures $G_{\theta}^+$ before and after the ADC.

**4. Nearly Gaussian Incoming Statistical State**

For the parameters explored in ref. 20, the incoming wave field is always characterized by a near-Gaussian distribution of the surface displacement. It is found that a physically consistent Gibbs measure should take negative values in the inverse temperature parameter $\theta < 0$, where a proper distribution function and a decaying energy spectrum are generated (ref. 26 and SI Appendix, Section B.1 have explicit simulation results). The upstream Gibbs measure $G_{\theta}$ with $D^- = 1$ displays a wide parameter regime in $(\theta^-, E_0)$ with near-Gaussian statistics. In Fig. 1A. *Left*, the inflow skewness $\kappa^-_3$ varies only slightly with changing values of $E_0$ and $\theta^-$. The incoming flow PDF then can be determined by picking the proper parameter value $\theta^-$ in the near-Gaussian regime with small skewness. In contrast, the downstream Gibbs measure $G_{\theta}^+$ with $D^+ = 0.24$ shown in Fig. 1A. *Right* generates much larger skewness $\kappa^+_3$ as the absolute value of $\theta^+$ and the total energy level $E_0$ increases. The solid lines in Fig. 1C offer a further confirmation of the transition from near-Gaussian statistics with small $\kappa^-_3$ to a strongly skewed distribution $\kappa^+_3$ after the depth change.

In the next step, the value of the downstream $\theta^+$ is determined based on the matching condition [4]. The expectation $\langle H_\Lambda \rangle_{G_{\theta}^+}$ about the incoming flow Gibbs measure can be calculated according to the predetermined parameter values of $\theta^-$ as well as $E_0$ from the previous step. For the direct numerical experiments shown later in Fig. 2, we pick proper choices of test parameter values as $L_0 = 6$, $E_0 = 100$, and $\theta^- = -0.1, -0.3, -0.5$. More test cases with different system energy $E_0$ can be found in SI Appendix, Section B.2, where similar transitions from near-Gaussian symmetric PDFs to skewed PDFs in the flow state $u_\Lambda^\pm$ can always be observed.

**Direct Numerical Model Simulations.** Other than the prediction of equilibrium statistical measures from the equilibrium statistical approach, another way to predict the downstream model statistics is through running the dynamical model [1] directly. The TKdV equation is found to be ergodic with proper mixing property as measured by the decay of autocorrelations as long as the system starts from a negative inverse temperature state as described before. For direct numerical simulations of the TKdV equations, a proper symplectic integrator is required to
guarantee that the Hamiltonian and energy are conserved in time. It is crucial to use the symplectic scheme to guarantee the exact conservation of the energy and Hamiltonian, since they are playing the central role in generating the invariant measure and the statistical matching. The symplectic scheme used here for the time integration of the equation is the fourth-order midpoint method (27). Details about the mixing properties from different initial states and the numerical algorithm are described in SI Appendix, Section C.

5. Predicting Extreme Anomalous Behavior After the ADC by Statistical Matching

With the inflow statistics well described and the numerical scheme set up, we are able to predict the downstream anomalous statistics starting from the near-Gaussian incoming flow going through the ADC from $D_1 = 1$ to $D_1 = 0.24$. We first consider the statistical prediction in the downstream equilibrium measure directly from the matching condition. The downstream parameter value $\theta^*$ is determined by solving the nonlinear Eq. 4 as a function of $\theta^*$, \[ F(\theta^*) = \langle H_x^3 \rangle_{\theta^*} - \langle H_x^3 \rangle_{\theta_0} = 0. \] In the numerical approach, we adopt a modified secant method, avoiding the stiffness in the parameter regime (SI Appendix, Section B.2 shows the algorithm). The fitted solution is plotted in Fig. 1B as a function of the proposed inflow $\theta^-$. A nonlinear $\theta^- - \theta^*$ relation is discovered from the matching condition. The downstream inverse temperature $\theta^*$ will finally saturate at some level. The corresponding downstream skewness of the wave displacement $u_\Lambda$ predicted from the statistical matching of Gibbs measures is plotted in Fig. 1C. In general, a large positive skewness for outgoing flow $\kappa_3^+$ is predicted from the theory, while the incoming flow skewness $\kappa_3^-$ is kept in a small value in a wide range of $\theta^-$. Note that, with $\theta^- \sim 0$ (that is, using the microcanonical ensemble only with energy conservation), the outflow statistics are also near Gaussian with weak skewness. The skewness in the outflow statistics grows as the inflow parameter value $\theta^-$ increases in amplitude.

For a second approach, we can use direct numerical simulations starting from the initial state sampled from the incoming flow Gibbs measure $\mu_\theta$ and check the transient changes in the model statistics. Fig. 2 illustrates the change of statistics as the flow goes through the ADC. Fig. 2, row 1 plots the changes in the skewness and kurtosis for the state variable $u_\Lambda$ after the depth change at $t = 0$. The PDFs in the incoming and outgoing flow states are compared with three different initial inverse temperatures $\theta^-$. After the depth changes to $D_1 = 0.24$ abruptly at $t = 0$, both the skewness and kurtosis jump to a much larger value in a short time, implying the rapid transition to a highly skewed non-Gaussian statistical regime after the depth change. Furthermore, from Fig. 2, different initial skewness (but all relatively small) is set due to the various values of $\theta^-$. With small $\theta^- = -0.1$, the change in the skewness is not very obvious (Fig. 2, row 2 shows the incoming and outgoing PDFs of $u_\Lambda$). In comparison, if the incoming flow starts from the initial parameter $\theta^- = -0.3$ and $\theta^- = -0.5$, a much larger increase in the skewness is induced from the ADC. Furthermore, in the detailed plots in Fig. 2, row 3 for the downstream PDFs under logarithmic scale, fat tails toward the positive direction can be observed, which represent the extreme events in the downstream flow (Fig. 3 shows the time series of $u_\Lambda$).

As a result, the downstream statistics in final equilibrium predicted from the direct numerical simulations here agree with the equilibrium statistical mechanics prediction illustrated in Fig. 1. The predictions from these two different approaches confirm each other.

6. Analytic Formula for the Upstream Skewness After the ADC

A statistical link between the upstream and downstream energy spectra can be found for an analytical prediction of the skewness in the flow state $u$ after the ADC. The skewness of the state variable $u_j$ at one spatial grid point is defined as the ratio between the third and second moments

$$ \kappa_3 = \langle u_j^3 \rangle_{\mu} / \langle u_j^2 \rangle_{\mu}^3. $$

Now, we introduce mild assumptions on the distribution functions.
- The upstream equilibrium measure $\mu_-$ has a relatively small skewness so that

$$ \langle H_3 \rangle_{\mu_-} = \frac{1}{6} \int_{-\infty}^{\infty} \langle u_3 \rangle_{\mu_-} \, dx \equiv \epsilon; $$

Fig. 2. Changes in the statistics of the flow state going through the ADC. The initial ensemble is set with the incoming flow Gibbs measure with different inverse temperature $\theta^-$. (Row 1) Time evolution of the skewness and kurtosis. The ADC is taking place at $t = 0$. (Row 2) Inflow and outflow PDFs of $u_\Lambda$. (Row 3) The downstream PDFs fitted with gamma distributions with consistent variance and skewness (in log coordinate in y). (Row 4) Energy spectra in the incoming and outgoing flows.
The downstream equilibrium measure \( \mu_k \) is homogeneous at each physical grid point so that the second and third moments are invariant at each grid point:

\[
\langle u_j^2 \rangle_{\mu+} = \sigma^2 = \pi^{-1}, \quad \langle u_j^3 \rangle_{\mu+} = \sigma^3 \kappa_3 = \pi^{-2} \kappa_3^+. \]

Then, the skewness of the downstream state variable \( u_j^+ \) after the ADC is given by the difference between the inflow and outflow wave slope energy of \( u_j^+ \):

\[
\kappa_3^+ = \frac{3}{2} \pi^2 L_0^{-2} E_0^{-\frac{1}{2}} D^2 \int_{-\infty}^{\infty} \left[ \langle u_j^2 \rangle_{\mu+} - \langle u_j^3 \rangle_{\mu-} \right] dx + 3 \pi^2 \epsilon. \quad [5]
\]

The detailed derivation is shown in SI Appendix, Section B.2. In particular, the downstream skewness with near-Gaussian inflow statistics \( \epsilon \ll 1 \) is positive if and only if the difference of the incoming and outgoing wave slope energy is positive. This means that there is more small-scale wave slope energy in the outgoing state. As evidence, in Fig. 2, row 4, in all of the weak and strong skewness cases, the outflow energy spectrum always has a slower decay rate than the inflow energy spectrum, which possesses stronger energy in larger scales and weaker energy in the smaller scales.

In Fig. 1C, we compare the accuracy of the theoretical estimation [5] with numerical tests. In the regime with small incoming inverse temperature \( \theta^- \), the theoretical formula offers a quite accurate approximation of the third-order skewness using only information from the second-order moments of the wave-slope spectrum.

7. Key Features from Experiments Captured by the Statistical Dynamical Model

In this final section, we emphasize the crucial features generated by the statistical dynamical model [1] by making comparison with the experimental observations in ref. 20. As from the scale analysis displayed in Section 2, the theory is set in the same parameter regime as the experimental setup.

- The transition from near-Gaussian to skewed non-Gaussian distribution as well as the jump in both skewness and kurtosis observed in the experiment observations (figure 1 in ref. 20) can be characterized by the statistical model simulation results (Fig. 2, rows 1 and 2). Notice that the differences in the decay of third and fourth moments in the far end of the downstream regime from the experimental data are due to the dissipation effect in the flow from the wave absorbers that is not modeled in the statistical model here. The model simulation time series plotted in Fig. 3 can be compared with the observed time sequences from experiments (figure 1 in ref. 20). The downstream simulation generates waves with strong and frequent intermittency toward the positive displacement, while the upstream waves show symmetric displacements in two directions with, at most, small peaks in slow time. Even in the time series at a single location \( x = 0 \), the long-time variation displays similar structures.

- The downstream PDFs in experimental data are estimated with a gamma distribution in figure 2 in ref. 20. Here, in the same way, we can fit the highly skewed outgoing flow PDFs from the numerical results with the gamma distribution

\[
\rho(u; k, \alpha) = \frac{e^{-k \alpha^{-1}}}{\Gamma(k)} (k^{-1} \alpha^{-1} u)_{k-1} e^{-u}. \]

- The parameters \((k, \alpha)\) in the gamma distribution are fitted according to the measured statistics in skewness and variance: that is, \( \sigma^2 = k \alpha^2, \kappa_3 = 2/\sqrt{k} \). Additionally, the excess kurtosis of the gamma distribution can be recovered as \( \kappa_4 = 6/k \). As shown in Fig. 2, row 3, excellent agreement in the PDFs with the gamma distributions is reached in consistency with the experimental data observations. The accuracy with this approximation increases as the initial inverse temperature \( \theta^- \) increases in value to generate more skewed distribution functions.

- Experimental measurements of the power spectra (figure 4 in ref. 20) reveal the downstream measurements to contain more energy at small scales (i.e., a relatively slower decay rate of the spectrum). This result is also observed in the direct numerical
simulations here (detailed results are shown in SI Appendix, Section C.2), as the outgoing state contains more energetic high frequencies.

8. Concluding Discussion
We have developed a statistical dynamical model to explain and predict extreme events and anomalous features of shallow water waves crossing an ADC. The theory is based on the dynamical modeling strategy consisting of the TKE equation matched at the ADC with conservation of energy and Hamiltonian. Predictions can be made of the extreme events and anomalous features by matching incoming and outgoing statistical Gibbs measures before and after the abrupt depth transition. The statistical matching of the known nearly Gaussian incoming Gibbs state completely determines the predicted anomalous outgoing Gibbs state, which can be calculated by a simple sampling algorithm verified by direct numerical simulations, and successfully captures key features of the experiment. An analytic formula for the anomalous outgoing skewness is also derived. The strategy here should be useful for predicting extreme statistical events in other dispersive media in different settings.

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