Supernova, nuclear synthesis, fluid instabilities, and interfacial mixing

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Supernovae and their remnants are a central problem in astrophysics due to their role in the stellar evolution and nuclear synthesis. A supernova’s explosion is driven by a blast wave causing the development of Rayleigh–Taylor and Richtmyer–Meshkov instabilities and leading to intensive interfacial mixing of materials of a progenitor star. Rayleigh–Taylor and Richtmyer–Meshkov mixing breaks spherical symmetry of a star and provides conditions for synthesis of heavy mass elements in addition to light mass elements synthesized in the star before its explosion. By focusing on hydrodynamic aspects of the problem, we apply group theory analysis to identify the properties of Rayleigh–Taylor and Richtmyer–Meshkov dynamics with variable acceleration, discover subdiffusive character of the blast wave-induced interfacial mixing, and reveal the mechanism of energy accumulation and transport at small scales in supernovae.

Astrophysics and Fluid Dynamics

Our understanding of stellar evolution is based on the conventional assumption that stars may be treated as spherically symmetric objects, at least on average (1). This approximation, which has proven to be surprisingly successful, allows us to evolve stars up to a given condition within a supernova remnant (SNR) (2–4). Young SNRs still retain information concerning the explosion process. Explosions are initial value problems; solution requires the details of what explodes (1–5).

Supernovae are violent, disruptive explosions of stars (1). They have been a central problem in astrophysics since their discovery and identification in the 1930s. The debris ejected from a supernova mixes with the interstellar medium, forming a supernova remnant (SNR) (2–4). Young SNRs still retain information concerning the explosion process. Explosions are initial value problems; solution requires the details of what explodes (1–5).

Taylor and Richtmyer–Meshkov instabilities and leading to intensive interfacial mixing of materials of a progenitor star. Rayleigh–Taylor and Richtmyer–Meshkov mixing breaks spherical symmetry of a star and provides conditions for synthesis of heavy mass elements in addition to light mass elements synthesized in the star before its explosion. By focusing on hydrodynamic aspects of the problem, we apply group theory analysis to identify the properties of Rayleigh–Taylor and Richtmyer–Meshkov dynamics with variable acceleration, discover subdiffusive character of the blast wave-induced interfacial mixing, and reveal the mechanism of energy accumulation and transport at small scales in supernovae.

How this happens is of considerable interest (1–7). Some supernovae produce neutron stars (pulsars), and others produce black holes. They are thought to be the major source of galactic cosmic rays. Supernovae are the dominant source of elements not produced by the Big Bang (those being hydrogen and helium). The calcium in our bones and the iron in our blood were synthesized in a supernova as were silver, gold, uranium, and thorium. The latter elements can also be produced in neutron star mergers in the "r-process," evidenced in the mergers' observations and the abundances in dwarf galaxies (1–11). Even carbon and nitrogen are partially produced in supernovae but more so in

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stars that die less violently as are the “s-process” heavy elements. The details are still in debate. Supernovae are a major source of radioactive nuclei for meteorites and for the early solar system (1–7).

Fig. 1 provides a detailed look at the Cassiopeia A (Cas A) SNRs, the youngest nearby SNR known in the Milky Way. The SNRs of the Cas A have been produced by the explosion of a massive star. The image in Fig. 1 is combined from 18 images taken by NASA’s Hubble Space Telescope in 2004. It shows the Cas A remnant as a broken shell of filamentary and clumpy stellar ejecta glowing with the heat generated by the passage of a shock wave from the supernova blast. The various colors of the gas indicate differences in chemical composition. Bright green filaments are rich in the oxygen, red and purple ones are rich in the sulfur, and blue ones are composed mostly of the hydrogen and the nitrogen. The oxygen and the sulfur are produced by thermonuclear burning during and just before the explosion. The sulfur is formed by the oxygen nuclear burning. Some of the sulfur lies farther from the center than some of the oxygen, which is interpreted as due to Rayleigh–Taylor instabilities (RTIs) during the explosion (12–14).

Hence, the key question is: can we work backward from the SNR toward the underlying explosion to provide insight into the event that is independent of conventional stellar evolution theory? To do so, we have to better understand fluid dynamic aspects of the multiphysics problem of supernovae, particularly Rayleigh–Taylor instability, Richtmyer–Meshkov instability (RMI), and Rayleigh–Taylor (RT)/Richtmyer–Meshkov (RM) interfacial mixing that are caused by supernova’s blast (1, 12–16). RTI develops at the interface of the fluids with distinct densities that are accelerated against their density gradients (12–14). RMI develops when the acceleration is induced by a shock refracting a perturbed interface of the fluids with distinct acoustic impedances (17, 18). Intense interfacial RT/RM mixing of the fluids ensues with time (12–18).

RTI/RMI and RT/RM mixing occur at a broad range of astrophysical phenomena in low- and high-energy density regimes (1, 14). Examples include the appearance of stiff light years-long structures in molecular hydrogen clouds, the formation of accretion disks and black holes, and the processes of stellar evolution (1–7). The latter ranges from a birth of a star due to the interstellar gas collapse to life of a star with the extensive material mixing in the stellar interior and to death of a star in the supernova (1–7, 14, 15). In supernovae, the blast wave-driven RT/RM mixing of the outer and inner layers of the progenitor star creates conditions for synthesis of heavy and intermediate mass elements in addition to light mass elements synthesized in the star before its explosion (1–7, 14, 15, 19–25).

In everyday life and in extreme astrophysics environments, RT flows are observed to have similar qualitative features of their evolution (12–16). RTI starts to develop when the fluid interface is slightly perturbed near its equilibrium state (12–14). The flow transits from an initial stage, where the perturbation amplitude grows quickly, to a nonlinear stage, where the growth rate slows and the interface is transformed into a composition of small-scale shear-driven vortical structures and a large-scale coherent structure of bubbles and spikes [with the bubble (spike) being the portion of the light (heavy) fluid penetrating the heavy (light) fluid]. RT flows are usually three-dimensional (3D) and have two macroscopic-length scales—the amplitude in the acceleration direction and the spatial period or the wavelength in the normal plane. The flow eventual stage is the self-similar interfacial mixing (12–15).

While RT and RM flows are similar in many regards, there are also important distinctions (12–18, 26–29). For instance, postshock RM dynamics is a superposition of two motions (17, 18, 26–29). These are the background motion of the fluid bulk and the growth of interface perturbations (26–29). In the background motion, both fluids and their interface move as whole unit in the transmitted shock direction; this motion occurs even for an ideally planar interface, and is supersonic for strong shocks. The growth of the interface perturbations is due to impulsive acceleration by the shock; it develops only for a perturbed interface (26–29). The rate of this growth is subsonic, and the associated motion is incompressible. The growth rate is constant initially and decays with time later. RM unstable interface is transformed to a composition of a large-scale structure of bubbles and spikes and small-scale shear-driven vortical structures. Small-scale nonuniform structures also appear in the bulk (26–29). Self-similar RM mixing develops, and the energy supplied initially by the shock gradually dissipates (17, 18, 26–29).

In RT mixing with constant acceleration, the length scale, the velocity scale, and the Reynolds number increase with time (14–16). At a first glance, such flow should quickly proceed to a fully disordered turbulent state (30). Turbulence is a stochastic process insensitive to deterministic conditions with intense energy transport from the large to the small scales (30–32). In supernova environments, such transport may supply to small scales some activation energy required for the synthesis of heavy mass elements (1). Recent advances in the theory and experiment of RTI and RT mixing have found, however, that, in the high- and low-energy density regimes, the properties of heterogeneous, anisotropic, nonlocal, and statistically unsteady RT mixing depart from those of homogeneous, isotropic, local, and statistically steady canonical turbulence (14–16, 30–34). High Reynolds number, while necessary, is not a sufficient condition for turbulence to occur. RT mixing exhibits more order and has stronger correlations, weaker fluctuations, and stronger sensitivity to deterministic conditions compared with canonical turbulence (14–16). For RM mixing, strong sensitivity to deterministic conditions has also been
found along with its nominally large Reynolds number and small-scale interfacial vortical structures (26–29).

In supernovae in a hydrodynamic approximation, accelerations are induced by blast waves (1). Blast waves can be viewed as strong variable shocks (1, 19–25, 33). Blast wave dynamics is self-similar, and the blast wave-induced acceleration is a power law function of time (spatial coordinate) (1, 19–25, 33). Several questions thus appear. What are the properties of RTI with variable acceleration, and how do they differ from those of RTI with constant acceleration and from those of shock-driven RMI? How can these properties be applied for interpretation of astrophysical data? What are their potential outcomes for stellar evolution and nuclear synthesis?

In this work, we consider hydrodynamic aspects on the multiphysics problem of supernovae and their remnants. We focus on the dynamics of RTI and RT mixing with variable acceleration in a broad parameter regime (Figs. 2–7). The acceleration is a power law function of time (spatial coordinate). We find that, for variable acceleration, RT/RRM dynamics is multiscale and has two macroscopic-length scales—the amplitude in the acceleration direction and the spatial period in the normal plane (14–16, 34). Depending on the exponent of the acceleration power law, the dynamics can be RT type or RM type. For RT type, the acceleration sets the timescale at early stage and defines the nonlinear dynamics and the interfacial mixing at later stages. For RM type, the initial growth rate sets the timescale at early stage; at late stages, the drag defines the nonlinear dynamics and the interfacial mixing, and the initially supplied energy gradually dissipates (33, 35). The critical values of the exponent at which the transition occurs from RT- to RM-type dynamics are distinct for the linear, nonlinear, and mixing regimes. Particularly for blast wave-induced accelerations, the linear and nonlinear dynamics are RT type, and the mixing is RM type, but RM-type mixing develops quicker than the acceleration prescribes (33, 35). While for subdiffusive mixing dynamics, superdiffusive canonical turbulence may be a challenge (33, 35). Blast wave dynamics is self-similar, and the initially supplied energy gradually dissipates (33, 35).

Fluid Instabilities and Interfacial Mixing

Governing Equations. Dynamics of ideal fluids is governed by the conservation of mass, momentum, and energy:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P + \rho \mathbf{g},$$

where \(\mathbf{v}\) is the velocity; \(\rho\) is the density; and \(\mathbf{g}\) is the acceleration (17, 18, 25).

Acceleration \(\mathbf{g}\), \(|\mathbf{g}| = g\), is directed from the heavy to the light fluid. Initial conditions include initial perturbations of the fluid fields (12–18, 25–29, 34). For ideal fluids, the initial conditions set the length-scale \(\lambda\) and the timescale \(\tau\). Here, \(\lambda\) is the perturbation wavelength (spatial period), and \(\tau \sim \nu / g_0\) in case of constant acceleration \(g_0\). In realistic fluids, small scales are usually stabilized, and a characteristic scale \(\lambda_0\) corresponds to a fastest growing mode \(i.e., \lambda_0 \sim (\nu^2 / g_0)^{1/3}\), where \(\nu\) is the kinematic viscosity (37–39). The ratio of the fluids’ densities and the density jump at the interface is parameterized illustrated by numerical simulations of strong shock-driven RMI (17, 18, 25, 29, 36). Such effects should be considered in the interpretation of observational data (1). Technical details of this work are given in SI Appendix for corresponding sections.

Fig. 2. One parameter family of regular asymptotic solution for 3D flow with group p6mm at some Atwood numbers. RT/RRM-type nonlinear dynamics: the bubble velocity vs. (A and C) the bubble curvature and (B and D) the interfacial shear.

Fig. 3. Qualitative velocity field in laboratory reference frame near the bubble tip for incompressible immiscible ideal fluids at some Atwood number for a nonlinear solution in RT/RRM family in (A) the volume and (B) the plane, with the interface marked by a dashed curve.

\[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) + \nabla \cdot (\nabla \cdot \mathbf{v}) = 0, \quad \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P + \rho \mathbf{g},\]

\[\frac{\partial E}{\partial t} + \nabla \cdot (E \mathbf{v}) = 0,\]

where \(x_i\) are the spatial coordinates with \((x_1, x_2, x_3) = (x, y, z); t\) is time; \((\rho, \mathbf{v}, P, E)\) are the fields of density \(\rho\), velocity \(\mathbf{v}\), pressure \(P\), and energy \(E = \rho (e + \mathbf{v}^2 / 2)\); and \(e\) is the specific internal energy (25). The latter refers to energy per unit mass contained within a system, excluding the kinetic and the potential energy of the system as a whole (25). For immiscible fluids, the fluxes of mass, momentum, and energy obey the boundary conditions at the interface:

\[\mathbf{v} \cdot \mathbf{n} = 0, \quad |\mathbf{P}| = 0, \quad |\mathbf{W}| = \text{any}, \quad |\mathbf{W}| = \text{any},\]

where […] denotes the jump of functions across the interface; \(\mathbf{n}\) and \(\mathbf{r}\) are the normal and tangential unit vectors of the interface with \(\mathbf{n} = \nabla \theta / |\nabla \theta|\) and \(\mathbf{n} \cdot \mathbf{r} = 0\); and \(\theta = \theta(x, y, z, t)\) is a scalar function of the interface and with \(\theta > 0\) in the bulk of the heavy (light) fluid marked with subscript h(l). The specific enthalpy is \(W = e + P / \rho\). In a spatially extended system, the flow can be periodic in the plane \((x, y)\) normal to the z direction of gravity \(g\) and has no mass sources:

\[\mathbf{v}|_{z=+\infty} = 0, \quad \mathbf{v}|_{z=-\infty} = 0.\]
by the Atwood number $A = (\rho_h - \rho_l)/(\rho_h + \rho_l)$, $0 < A < \infty$; $A \to 1$ for $\rho_h/\rho_l \to \infty$ and $A \to 0$ for $\rho_h/\rho_l \to 1$ (12–18).

To rigorously describe RT dynamics, one has to solve the problem of extreme complexity: solve a system of nonlinear partial differential equations in 4D space–time, solve the boundary value problems for a subset of nonlinear partial differential equations at a nonlinear freely evolving interface and at the outside boundaries and also solve the ill-posed initial value problem, with account for singularities and secondary instabilities developing in a finite time (14, 34). A complete theory of RTI applicable at all scales and all times has yet to be developed. Rigorous theories have successfully handled the problem in well-defined approximations; empirical models have repeatedly described a broad set of data with nearly the same set of parameters. The reader is referred to review and research papers (12–18, 26–30, 33–54) for details of theoretical and numerical studies of RT dynamics.

It is worth noting that, despite complexity and noisiness resulting from interactions of all of the scales, RT dynamics is observed to have certain features of universality and order, and is thus eligible to first principle considerations, such as group theory (14, 15, 34, 55, 56). For linear and nonlinear RTI, group theory analysis uses theory of discrete groups to solve the boundary value and initial value problems (34, 48–50). For RT mixing, group theory is implemented in the momentum model with equations that have the same symmetries and scaling transformations as the governing equations (14, 16, 34, 40, 54). Some principal results of group theory analysis—the multiscale character of nonlinear dynamics, to which both the spatial period $\lambda$ and the amplitude $h$ contribute; the tendency to keep isotropy in the plane normal to the acceleration and the discontinuous dimensional cross-over; the order in RT mixing that may be coexistent with a quasiturbulent state—self-consistently explain the observations (14, 15, 26–50). For RTI mixing, group theory finds that, in RT mixing, the momentum and energy are gained and lost at any scale; the dynamics of a parcel of fluid is governed by a balance per unit mass the rates of momentum gain, $\bar{\mu}$, and momentum loss, $\mu$, as

$$\bar{h} = \nu, \quad \bar{\nu} = \bar{\mu} - \mu,$$

where $\bar{h}$ is the length scale along the acceleration $\bar{g}$, $\nu$ is the corresponding velocity, and $\bar{\mu}(\mu)$ is the magnitude of the rate of gain (loss) of specific momentum in the acceleration direction (14, 16, 54). The rate of gain (loss) of specific momentum is $\bar{\mu} = \varepsilon/\nu (\mu = \bar{\nu}/\bar{\nu})$, with $\varepsilon(\varepsilon)$ being the rate of gain (loss) of specific energy. The rate of energy gain is $\varepsilon = f g v$, $f = f(A)$, rescaled $g f \to g$ hereafter. The rate of energy dissipation is $\varepsilon = C v^3/L$, with a length scale $L$ and a drag $C$, $C \in (0, \infty)$ (14–16, 35, 54). Momentum model has the same symmetries and scaling transformations as the governing equations (14, 16, 34, 54). It can be solved by applying the Lie groups. The cases $L \sim \lambda$ and $L \sim h$ correspond to the nonlinear dynamics and the self-similar mixing (14–16, 34, 54). In each case, asymptotically, there is the particular solution of RT type ($h_\nu, v_\nu$) and the
homogeneous solution of RTM type \( (h_{ct}, v_0) \) (35). These solutions are effectively decoupled due to their distinct symmetries.

**Smoothed Particle Hydrodynamics Simulations.** Supernova environments are characterized by the conditions of high energy density, strong shocks, sharp changes of flow fields, large perturbations, and small effects of dissipation and diffusion. Numerical modeling of these extreme regimes is a challenge. This is because numerical methods should satisfy numerous competing requirements to capture shocks, track interfaces, and accurately account for dissipative processes. An efficient approach for modeling strong shock-driven dynamics is the smoothed particle hydrodynamics (SPH) implemented in the smoothed particle hydrodynamics code (SPHC) (36).

The SPH is a Lagrangian method representing a continuous fluid by means of fixed mass SPH particles and thus, reducing the governing partial differential equations to ordinary differential equations (26–29, 36). SPHC has been originated in astrophysics, and in addition to astrophysical gravitational problems, it has been successfully applied, tested, and validated in multiphysics problems in fluids, plasmas, materials for modeling the strong shock-driven RMI, the Noh problem, and other flow problems as well as reactive and supercritical fluids, material transformation under impact, ablation process in hypersonic flows, and charge imbalance in plasmas (36). When applied to strong shock-driven RMI, it has achieved an excellent agreement with the experiments and with the rigorous zero-order, linear, and group theories. The latter includes the multiscale character of the nonlinear dynamics and is evinced, for instance, in the flattening of RM bubbles, flow fields structure, and sensitivity to deterministic conditions (26–29, 36).

Here, we apply group theory to study RT/RMI and RT/RM mixing with variable acceleration in a broad parameter regime. We further conduct SPHC simulations of strong shocks-driven RM flows to study their sensitivity to deterministic conditions and their small-scale dynamics at the interface and in the bulk. We find the properties of blast wave-induced RT/RM dynamics and propose the mechanisms for energy accumulation and transport at small scales in supernovae.

**RTI with Variable Acceleration**

We consider RT dynamics with time-varying acceleration \( g = Gt^a \), where \( a \) is the acceleration exponent, \( a \in (-\infty, +\infty) \), and \( G \) is the prefactor, \( G > 0 \), with dimensions \( [G] = m/s^{a+2} \) and \( [a] = 1 \). For a given wavelength (period) \( \lambda \) and for \( a \in (-\infty, -2) \cup (-2, +\infty) \), there are two timescales \( \tau_G = (kG)^{-\frac{(a+2)}{2}} \) and \( \tau_0 = (kv_0)^{-1} \), where \( k \) is the wave vector with \( k = 4\pi/\sqrt{3} \) for 3D flow with group \( \rho_0m \) and \( v_0 \) is the initial growth rate set by the initial conditions and/or by the impulse acceleration. At \( a = -2 \), the timescale is \( \tau_0 = (kv_0)^{-1} \), and value \( G \) characterizes the acceleration strength. Time is \( t > 0 \).

At early stage, for \( a > -2 \), the timescale is \( \tau_G \), and the linear dynamics is driven by the acceleration and is RT type. For \( a < -2 \), the timescale is \( \tau_0 \), and the linear dynamics is driven by the initial growth rate and is RM type. At \( a = -2 \), the timescale is \( \tau_0 = (kv_0)^{-1} \), and the linear dynamics changes its character from RT to RM type with the decrease of \( Gk \).

At late stage, for \( a > -2 \), the nonlinear dynamics is RT type; regular asymptotic solutions depend on time as \( \zeta \sim k \) and \( v, M, M \sim t^{a/2} \) for \( t \gg \tau_G \). For \( a < -2 \), the nonlinear dynamics is RM type; regular asymptotic solutions depend on time as \( \zeta \sim k \) and \( v, M, M \sim t^{-1} \) for \( t \gg \tau_0 \). At \( a = -2 \), regular asymptotic solutions are \( \zeta \sim k \) and \( v, M, M \sim t^{-1} \) for \( t \gg \tau_0 \), and they are RT (RM) type for \( Gk > 1 \). These regular asymptotic solutions form a family (34).

For \( a > -2 \) and at \( Gk > 1 \), for regular asymptotic solutions of RT-type nonlinear dynamics, the bubble velocity \( v \geq 0 \) depends on its curvature \( \zeta \), \( \zeta < 0 \), as

\[
\nu \sqrt{\frac{-4A(\zeta/k)^2 + 9 + 64(\zeta/k)^2}{4(\zeta/k)^2}}^{-1}.
\]

For every \( A \), this function domain is \( \zeta \in (\zeta_{cr}, 0), \zeta_{cr} = -(3/8)k \), and the range is \( \nu \in (0, \nu_{\max}) \), with \( \nu = \nu_0 \) achieved at \( \zeta = 0 \) and at \( \zeta = \zeta_{cr} \) with \( \nu = \nu_{\max} \) achieved at \( \zeta = \zeta_{\max} \), \( \zeta_{\max} \in (\zeta_{cr}, 0) \).

The multiplicity of the nonlinear solutions is associated with the nonlocal and singular character of the interfacial dynamics (34, 45, 46, 48–50). The solutions exist and converge with increase in approximation order. The number of the family parameters is identified by symmetry of the global flow (34). For group \( \rho_0m \), the dynamics is highly isotropic, \( z^* - z_0 \sim \zeta(x^2 + y^2) \), and the interface morphology is captured by the principal curvature \( \zeta \) (Fig. 2) (34, 48–50).

The multiplicity is also due to the presence of shear at the interface (58). Defining the shear as the spatial derivative of

![Fig. 6. Wave interference and order–disorder in RM flow. Snapshots of the flow regions at some time instances for two-wave initial perturbations, with the same waves being in (Left) antiphase and (Right) random phase.](Image)

\( \tau_{G0} = 5/3, M = 5, A = 0.8, \lambda/v_0 = 0.86 \mu s, t = 1.4 \mu s \)

\( \tau_{G0} = 5/3, M = 5, A = 0.8, \lambda/v_0 = 0.86 \mu s, t = 1.4 \mu s \)

**Fig. 7. Small-scale nonuniform structures in the fields of temperature (Left) and pressure (Right) in RM flow, including bulk-immersed cumulative jets, hot (cold) spots, and high- (low-) pressure regions. Reprinted from ref. 27, with permission of AIP Publishing.**

\( A\bar{b}r\z\tilde{e}z. \)
the jump of tangential velocity at the interface, $\Gamma = \Gamma(\xi, y)$, with $\Gamma_0(\xi, y) = \partial_y \nu(y) |_{\Gamma}$, we find shear $\Gamma = M_1 - M_2$ near the bubble tip, and its dependence on the interface curvature $\zeta$. For $\zeta \in (-\zeta_\text{cr}, 0)$, shear $\Gamma = 1$–1 function on $\zeta$. $\Gamma \in (\Gamma_{\text{min}}, \Gamma_{\text{max}})$ achieving value $\Gamma_{\text{max}}$ at $\zeta = 0$ and value $\Gamma_{\text{min}}$ at $\zeta = \zeta_{\text{cr}}$.

The fastest stable solution is the physically significant solution (Fig. 2) (14, 34, 48–50, 58). For the fastest stable solutions, the dependence of curvature and velocity ($\zeta_{\text{max}}$, $\nu_{\text{max}}$) on the Atwood number $A$ is complex: $\nu_{\text{max}} = \nu_{\text{max}}(G^2, k, A)$, $\zeta_{\text{max}} = \zeta_{\text{max}}(k, A)$ (Fig. 2). However, there is the invariant $\nu_{\text{max}}^2 = (t^2 G/k) (B_{\text{cr}}(k, A) / k)^3 = 1$, implying that the nonlinear dynamics is characterized by the contribution of the wavelength $\lambda$, the amplitude $\zeta$, and their derivatives and is, thus, multiscale (34, 48–50).

Regular asymptotic solutions have important physics properties. For these solutions, there is effectively no motion of the fluids in the bulk and away from the interface, there is intense motion near the interface, and shear is present at the interface leading to formation of interfacial vortical structures, in agreement with observations (Fig. 3) (34, 48–50).

Regular asymptotic solutions have important global properties. The flow tends to conserve isotropy in the plane (34, 48–50). The 3D highly symmetric dynamics is universal (34, 48). That is, on the substitution $k = 2x/\lambda$, the nonlinear solutions describe the dynamics of 3D flow with group p4mm. For 3D low-symmetric flows with group p2mm, there is a two-parameter family of regular asymptotic solutions; among the family solutions, only nearly isotropic bubbles are stable. The dimensional 3D–2D cross-over is discontinuous (34, 48).

At $a = -2$ with $Gk < 1$ and for $a > -2$, for regular asymptotic solutions of RTM-type nonlinear dynamics, the bubble velocity $\nu \geq 0$ depends on the bubble curvature $\zeta$, $\zeta < 0$, as

$$
\nu(kt) = \left(3 - 2A(z/k) \left(-5 + 64(z/\zeta)^2\right)\right) \left(9 - 64(z/\zeta)^2\right) \left(-48(z/\zeta) + A \left(9 + 64(z/\zeta)^2\right)^{-1}\right).
$$

The function domain is $\zeta \in (-\zeta_{\text{cr}}, 0)$, $\zeta_{\text{cr}} = (3/8)k$, and the range is $\nu \in (\nu_{\text{min}}, \nu_{\text{max}})$, with $\nu = \nu_{\text{min}}$ achieved at $\zeta = \zeta_{\text{cr}}$ and $\nu = \nu_{\text{max}}$ achieved at $\zeta = 0$. For $\zeta \in (\zeta_{\text{cr}}, 0)$, $\nu_{\text{min}} \geq \nu_{\text{cr}}$, shear $\Gamma$ is 1–1 function on $\zeta$, $\Gamma \in (\Gamma_{\text{min}}, \Gamma_{\text{max}})$, with $\Gamma = \Gamma_{\text{max}}$ at $\zeta = \zeta_{\text{cr}}$ and $\Gamma = \Gamma_{\text{min}}$ at $\zeta = 0$. The fastest stable solution is the physically significant solution (Fig. 2) (14, 34, 48–50, 58).

RT and RM families have similar physical, mathematical, and global properties (Figs. 2 and 3). These include the structure of the velocity fields, the existence of the family of solutions, their dependence on the flow symmetry and the interface shear, the tendency to keep isotropy in the plane normal to the acceleration, and the discontinuity of the 3D–2D cross-over. However, their local bubbles, the velocity is a monotone function on the curvature, and the fastest stable solution corresponds to a flat bubble with $\nu_{\text{max}}(\nu_{\text{max}}(kt) / 3) = 1$, $\zeta_{\text{max}} = 0$. The quasi invariant of this solution $(4/3) \nu_{\text{max}}^\sigma / (\partial y / \partial x) |_{z = \zeta_{\text{max}}} = (1 + (5/2)(A/2)^{-1}) \sim 1$ implies that the wavelength and the amplitude both contribute to the nonlinear dynamics. A steep dependence of the velocity on the shear for physically significant solutions in RT/RM families suggests the use of highly accurate methods with the interface tracking for numerical modeling RT/RM dynamics (Figs. 2 and 3).

### Mixing Dynamics

**RTM Dynamics for Time-Varying Acceleration.** In the nonlinear regime with $L = \lambda$, asymptotic solutions for the momentum model are consistent with group theory results. In the mixing regime with $L \sim h$, asymptotic solutions for the momentum model are $h_2 = B_2 t^{a_2}$ and $h_2 = B_2 t^{a_2}$, where the critical exponent is $a_2 = -2 + (1 + C)^{-1}$ with $a_2 \rightarrow -1$ for $C \rightarrow 0$ and $a_2 \rightarrow -2$ for $C \rightarrow \infty$ (33). For solution $(h_2, \nu_2)$, the exponent is set by the acceleration’s exponent, $(2 + a)$, and the prefactor $B_2$ is set by the acceleration parameters and the drag. This mixing is RT type; it is driven by the acceleration. For solution $(h_2, \nu_2)$, the exponent is set by the drag (the dissipation, which $\nu = e/\nu$) (35). For $a > a_2$ the mixing is RT type. For $a < a_2$, the mixing is RM type. At $a = a_2$, a transition occurs from RT- to RM-type mixing (33, 35).

**RT-Type Mixing.** Properties of the asymptotic solutions indicate that, for $a > a_2$, when the dynamics is the acceleration driven in the nonlinear and mixing regimes, two states are possible. One state is achieved for $L \sim L$, the state with asymptotic balance of the rates of momentum, $\mu \sim |\mu| \sim t$, and $\nu \sim |\nu| \rightarrow 0$ (14, 33, 54). The other is achieved for $L \sim h$; it is the state with $\mu \sim |\mu| \sim |\nu| \sim t$ and with an algebraic imbalance of the rates of momentum, $\mu \neq |\mu|$ (13, 33, 54). Per observations, at $a = 0$, the imbalance is small, $\mu \sim |\mu| / \mu \ll 51–53$. Hence, RT mixing may develop when the amplitude $h$ is the scale for energy dissipation, $L \sim h$ (14, 33, 35). It may also develop due to the growth of period $\lambda \sim Gt^{\mu + 2}$ in the nonlinear regime. Because the dynamics is multiscale, the growth of the period $\lambda$ is possible and is not a necessary condition for the mixing to occur (14, 29, 33, 35, 54).

For $a > a_2$, in RT-type mixing, the length scales with time as $L \sim t^{a_2}$, and the velocity scales as $v \sim t^{a_2 - 1}$. The length scale increases with time for $a > a_2$. The velocity scale increases for $a > a_2$, is constant for $a = -1$, and decreases for $a < -1$. Recall that diffusion scaling law is $L \sim t^{1/2}$, whereas canonical turbulence is superdiffusive with $L \sim t^{1/2}$ (25, 31, 32, 59). By comparing these exponents, we find that, in RT-type mixing with $a > a_2$, the dynamics $L \sim t^{a_2}$ is super ballistics for $a > 0$, ballistics at $a = 0$, steady at the flex point $a = -1$, superdiffusion for $a > -3/2$, quasidiffusion at $a = -3/2$, and subdiffusion for $-3/2 > a > a_2$. Large velocities correspond to small (small) length scales for $a > -1$ ($a_2 < a < -1$).

**RM-Type Mixing.** RM-type mixing develops for $a < a_2$; its rates of gain and loss of momentum are asymptotically imbalanced $|\mu| \sim \mu| \rightarrow 0$, whereas $|\mu| \sim |\nu| \sim h_3$ (33, 35). RM-type mixing may develop when the amplitude $h$ is the scale for energy dissipation, $L \sim h$. It may also develop when the period $\lambda$ grows with time, as $\lambda \sim (t^{a_2})^{1 + (a_2 - a)}$ (13, 33, 54) for $-2 < a < a_2$. In RTM-type mixing with $a < a_2$, the length scale increases with time, $L \sim t^{a_2}$, whereas the velocity scale $v \sim t^{a_2 + 1}$ decreases with time, and large velocities correspond to small length scales, since $(a_2 + 2) \in (-1, 0)$ for $C > 0$.

**Space-Varying Acceleration.** Similar analyses can be conducted when the acceleration is a power law function on the spatial coordinate, $g \sim h^n$. Particularly, the nonlinear dynamics is RT type for $n \in (-\infty, 2)$ with $v \sim t^{(a_2 + 1 - n)}$, and RM type for $n = -\infty$ with $v \sim t^{-1}$. For $n \rightarrow -\infty$, the nonlinear dynamics changes from RT to RM type.
Solutions for the nonlinear dynamics with time- and space-varying accelerations can be transformed one into another with $\alpha \to 2n/(2 - n)$.

The mixing dynamics is RT type for $n \in (n_c, 1)$ with $h \sim t^{2(1-n)/n}$, and RM type for $n \in (-\infty, n_c)$ with $h \sim t^{2(1-n)/n}$. The critical exponent is $n_c = -2C - 1$, with $n_c \to -1$ for $C \to 0$ and $n_c \to -\infty$ for $C \to +\infty$. Solutions for the mixing dynamics with time- and space-varying accelerations can be transformed into one another with $\alpha \to 2n/(2 - n)$.

Fig. 4 illustrates RT/RM-type mixing for various values of the exponents $\alpha$ and $n$. Solutions are derived for $C$ being a stochastic process with log-normal distribution with the mean $\langle C \rangle = 3.6$ and the SD $\sigma = (|C|/2)$ leading to $a_{cr} \approx -1.78$ ($n_c \approx -8.2$). Mean values of quantities $h, v, \sqrt{g}$ are plotted in red, blue, green, and black, respectively, in Fig. 4 (35).

**Blast Wave-Induced Mixing.** Consider now RT/RM mixing induced by blast waves with the first kind (Sedov–Taylor) self-similarity and the second kind (Guderley–Stanyukovich–Landau) self-similarity (19–25). Note that exponents $a_{cr}(n_c)$ have values typical for blast waves.

For Sedov–Taylor self-similar dynamics, the scaling dependence of the solution is set by the energy release $E$, $|E| = kg(m/s)^3$, and the fluid density $\rho$, $|\rho| = kg/m^2$, in 3D (2D, 1D) in case of point (line, plane) energy source (19, 20, 24). The invariance of energy density $E/\rho$ leads to scaling laws for the length $\sim t^{2/5}(t^{1/2}, t^{2/3})$, velocity $\sim t^{-3/5}(t^{-1/2}, t^{-1/3})$, and acceleration $\sim t^{-8/5}(t^{-3/2}, t^{-4/3})$. This suggests that the blast wave dynamics is substantially slower than canonical turbulence (31, 32). By comparing the blast wave acceleration exponent with the value $a_{cr}$, we find that this mixing can be RT type for small drag $C < 3/2(1, 1/2)$ and RM type for large drag $C > 3/2(1, 1/2)$.

For Guderley–Stanyukovich–Landau self-similar dynamics, the load history and momentum transport should be accounted for. The corresponding invariant $|F(E, P, \rho)|$ is a function of the energy release $E$, pressure $P$, and fluid density $\rho$, with $|F| \approx m/s^2$ and $0 < \theta < 1$ (21–24). In this case, the solution is a power law with the length $\sim t^\phi$, velocity $\sim t^{\phi-1}$, and acceleration $\sim t^{\phi-2}$, so that the acceleration exponent is $\sim -2(\theta - 2) < -1$. By comparing these values with $a_{cr} = -2 + (1 + C)^{-1}$, we find that this mixing can be RT type for small drag $C < (\theta^{-1} - 1)$ and RM type for large drag $C > (\theta^{-1} - 1)$.

**Small-Scale Nonuniform Structures of the Flow Fields.** Hence, depending on the drag value, the blast-induced mixing can be RT type with $a \sim a_{cr}$ or RM type with $a \leq a_{cr}$. In either case, larger velocities (turbulence fluctuations) correspond to small length scales, and the dynamics is sensitive to deterministic conditions. In canonical turbulence, large velocities (turbulence fluctuations) correspond to large length scales, and the dynamics is independent of deterministic conditions (31, 32). In the blast wave-induced (subdiffusive) mixing, the canonical (superdiffusive) turbulence may be a challenge to develop (unless there is a source, other than gravity, supplying turbulent energy to the fluid system). If so, what are other possible mechanisms for energy accumulation at small scales, which is necessary for nuclear synthesis in supernovae (1)?

Such mechanisms may exist due to small-scale nonuniform structures of the flow fields (26–29, 59). Particularly, subdiffusive processes are often characterized by localization. If a spot with high energy fluctuation appears in the flow, it can be trapped due to the slow dynamics. If the spot is hot enough, it may initiate a nuclear reaction accompanied by some energy release and may further induce a chain of other reactions and processes (1–3). Blast waves are the special strong shocks (24). In strong shock-driven RMI, the shock-interface interaction may lead to intense production of small-scale nonuniform structures in the bulk in addition to small-scale shear-driven vortical structure at the interface (26–29). These nonuniform structures may include cumulative jets, hot and cold spots, high- and low-pressure regions, and may enable strong energy fluctuations at small scales.

**Strong Shock-Driven RM Flows**

**SPH Simulations.** To illustrate scale coupling in strong shock-driven RMI, we use the SPH in a hydrodynamic approximation for ideal monoatomic gases with adiabatic indexes $\gamma_{	ext{H}_2} = 5/3$ and the Atwood number $A = (0.3, 0.6, 0.7, 0.8, 0.95)$. They have high energy per atom. The shock Mach number is $M = (3.5, 7.10)$ defined relatively to the light fluid with the speed of sound $c_1$. The interface is normal to the shock; the initial perturbation wavelength and amplitude are $\lambda$ and $a_0$. In SPH, we scale the length, velocity, and time with $\lambda, v_\infty, \lambda/v_\infty$, where $v_\infty(M, A, \gamma_{	ext{H}_2})$ is the background motion velocity; it is supersonic. RM initial growth rate $v_0(M, A, \gamma_{	ext{H}_2}, \lambda, a_0)$ is subsonic.

Fig. 5A illustrates SPH simulations of the postshock dynamics of strong shock-driven RMI. The superposition of the growth of the interface perturbation with the background motion of the fluids, the formation of large-scale coherent structure of bubbles and spikes, the bubble flattening at late times, and the occurrence at small scales of interfacial vortical structures and bulk-immersed cumulative jets are seen in Fig. 5A. The flow regions are shown, with red (blue) for the light (heavy) fluid particles and green for the light fluid interfacial particles in Fig. 5A.

**Sensitivity of RM Dynamics to Deterministic Conditions.** In RMI with a single-wave initial perturbation, the initial growth rate $v_0$ depends on $\lambda$ and $a_0$, whereas the nonlinear dynamics retains memory of the initial conditions (26–29). Fig. 5B shows the dependence of the initial growth rate on initial perturbation amplitude for given $(M, A, \gamma_{	ext{H}_2}, \lambda)$ and for $a_0/\lambda \in [0.1, 1]$. The data are confidently described by the model $(v_0/v_\infty)/A = C_1(a_0/\lambda)e^{-C_2(a_0/\lambda)}$ with $C_1 \approx 4.26$ and $C_2 \approx 2.63$ (Fig. 5 B and C), suggesting that, in addition to wavelength $\lambda$, RM dynamics has the characteristic amplitude scale $a_{\text{max}} \approx 0.38\lambda$, $a_{\text{max}} \approx 2.4$, at which the maximum initial growth rate is achieved, $v_0/|v_\infty| \approx 0.64$. The ratio of the RM initial and linear growth rates decays exponentially with the initial amplitude, $v_0/|v_\infty|_{\text{lin}} = e^{-\lambda(a_0/\lambda)}$ (Fig. 5) (28).

For a multimode initial perturbation, the order and disorder in RM flow are highly sensitive to deterministic conditions, including the wavelengths, the amplitudes, and the relative phases of the initial perturbation waves (29). Fig. 6 show snapshots of the flow regions of late stages of the RMI, with the same corresponding values of the Mach and Atwood numbers and the observational time. The initial perturbations have two waves with the same amplitudes and wavelengths $(\lambda_1, a_1, \lambda_2, a_2)$). However, the flow keeps order in some cases and is disordered in the others. This difference is due to the relative phase $\varphi$ and the interference of waves constituting the initial perturbation, as group theory finds (29). In Fig. 6, Left, the waves are antiphase, $\varphi = \pi$, and the flow symmetry group is pm1; this flow keeps order (29, 34). In Fig. 6, Right, the waves have random phase, $\varphi = \pi/2$, and the flow symmetry group is p1; this flow is disordered (29, 34). While at a first
Small-Scale Structures in RM Flow. SPHC simulations accurately capture small-scale dynamics in strong shock-driven RMI. At the interface, we observe small-scale shear-driven vortical structures due to the Kelvin–Helmholtz instability (26–29). In the bulk, between the transmitted shock and the interface, we observe small-scale nonuniform localized structures of the flow fields (Fig. 7) (26). These include the “reverse” cumulative jets, which are short and energetic and develop due to collisions of converging fluid flows in the bulk (56, 60); the hot (cold) spots, which are the localized regions with the temperature much higher (lower) than that of the ambient; and the regions with high (low) pressure (26–29). These well-pronounced small-scale structures are volumetric in nature (Fig. 7). They develop, since the flow dynamics is adjusted to the motion of the interface.

Hence, the strong shock-driven RM dynamics is sensitive to deterministic conditions, may keep order at large and small scales, and may have localized nonuniform structures at small scales in the bulk in addition to shear-driven vortical structures at the interface.

Discussion
Supernovae and their remnants are a central problem in astrophysics due to their role in the formation of neutron stars and black holes in the processes of stellar evolution and nuclear synthesis (Fig. 1) (1). We have considered this multiphysics problem in a hydrodynamic approximation, with supernova blast causing the development of RTI, RMI, and RT/RM interfacial mixing with variable acceleration (1–7). We have applied group theory (14, 34) to study RTI/RMI and RT/RM mixing with variable acceleration, and identified properties of linear, nonlinear, and mixing RT/RM dynamics that have not been discussed before (Figs. 2–7).

We have found that, for $g \sim t^a$, the linear and nonlinear dynamics is RT type and is acceleration driven for $a < -2$; for $a > -2$, the dynamics is RM type and is driven by the initial growth rate at early stage and by the drag (dissipation) at late stages. RT–RM transition occurs at $a = -2$ with varying of the acceleration strength. For any $a$, the nonlinear regular asymptotic solutions form a continuous family; this multiplicity is due to the interfacial shear, and the fastest stable solution is physically significant (Figs. 2 and 3). The principal result of the group theory analysis is the multiscale character of the nonlinear RT/RM dynamics, to which two macroscopic-length scales—spatial period and amplitude—contribute. This further leads to two distinct mechanisms of the development of RT/RM mixing: the growth of the wavelength and the dominance of the amplitude, each resulting in the imbalance of gain and loss of the rates of specific momentum (14, 34).

The mixing is RT type for $a > a_{cr}$, and is RM type for $a < a_{cr}$. RT–RM transition occurs for $a \sim a_{cr}$ by varying the acceleration exponent; the critical exponent $a_{cr} = -2 + (1 + C)^{-1}$, $a_{cr} \in (-2, -1)$ depends on the flow drag (Fig. 4). For $-2 < a < a_{cr}$, the asymptotic dynamics is RT type in the linear and nonlinear regimes and RM type in the mixing regime: The acceleration triggers the instability and defines the linear and nonlinear dynamics, but it plays effectively no role in the mixing regime. The mixing dynamics is nevertheless faster than the acceleration prescribes. The critical exponent $a_{cr}$ has the exponent values typical for blast waves.

For RT-type mixing with $a > a_{cr}$ and length-scale $L \sim t^{a_{cr}}$, with the decrease of the acceleration exponent, the dynamics changes its character from superballistics to subdiffusion. For RM-type mixing with $a < a_{cr}$ and length-scale $L \sim t^{a_{cr}}$, the dynamics is faster than the acceleration prescribes and is subdiffusive for $C > 1$. For $a > -1$ ($a < -1$), large velocities correspond to large (small) scales.

In blast wave-driven dynamics, the acceleration exponent can be $a \sim a_{cr}$ or $a < a_{cr}$ (19–25). The blast wave-driven mixing can thus be RT type with $a > a_{cr}$ or RT type with $a < a_{cr}$. In either case, the mixing has large velocities (velocity fluctuations) at small scales, and the dynamics is essentially subdiffusive. Subdiffusive processes are known to depend on deterministic conditions and have small-scale localizations (Figs. 5–7). RM dynamics is indeed sensitive to deterministic conditions, including the wavelengths, the amplitudes, the relative phase, and the interference of waves constituting the initial perturbation. In RM flows, localized nonuniform structures develop at small scales in the bulk in addition to shear-driven vortical structures at the interface. These volumetric structures may include cumulative jets, hot and cold spots, and high- and low-pressure regions (26–29). They are well-pronounced and cause strong fluctuations of flow fields. Depending on deterministic conditions, RM dynamics may keep order at large and small scales.

What are potential outcomes of this hydrodynamics for supernova and nuclear synthesis (1)?

In blast wave-driven RT/RM mixing in supernovae, canonical turbulence is traditionally considered as the mechanism for energy accumulation at small scales. According to our results, for the acceleration parameters typical for blast waves (19–25), superdiffusive turbulence (31, 32) may be a challenge to implement in subdiffusive RT/RM mixing. However, the conditions of heterogeneity, nonlocality, anisotropy, and statistical unsteadiness that are common for RT/RM flows (14, 15, 33, 34) may lead to appearance of small-scale nonuniform structures in the bulk of the blast wave-driven mixing flows, whereas the slow subdiffusive transport may result in energy localization and trapping at small scales (26–29, 59). These effects are consistent with and may explain the richness of structures observed in supernovae, including the Cas A (Fig. 1) (1). We suggest that such effects be considered in interpretation of observational data.

Our work focuses on hydrodynamic aspects of the multiphysics problem of supernovae and their remnants and serves for systematic studies of the problem in perspective.

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