Quantum Fourier analysis is a subject that combines an algebraic Fourier transform (pictorial in the case of subfactor theory) with analytic estimates. This provides interesting tools to investigate phenomena such as quantum symmetry. We establish bounds on the quantum Fourier transform $\mathcal{F}$, as a map between suitably defined $L^p$ spaces, leading to an uncertainty principle for relative entropy. We cite several applications of quantum Fourier analysis in subfactor theory, in category theory, and in quantum information. We suggest a topological inequality, and we outline several open problems.

picture language | uncertainty principles | quantum symmetries | inequalities | quantum entanglement

In this paper, we explore quantum Fourier analysis (QFA), a subject revolving around the study of Fourier analysis of quantum symmetries. The discovery of such symmetries emerged in the 1970s and has flourished ever since. It represents a major advance both in mathematics and in physics, as well as in the relation between these two subjects. Thus, QFA adds an extra dimension to the 200-year-old subject of classical Fourier analysis (CFA), analyzing the Fourier transform $F$.

CFA led to insights into and to solutions of problems in almost every field of mathematics, including partial differential equations, probability theory, number theory, representation theory, topology, geometry, etc. It ultimately led to the categorization of Fourier duality (1, 2). The Hausdorff–Young inequality is a bound on the norm $M_p = \|F\|_{L^p \to L^q}$, where $q = p/(p-1)$. Hirschman discovered that differentiating $M_p$ gives an uncertainty principle for the Shannon entropy, generalizing the well-known Heisenberg principle. He and Everett conjectured the optimal inequality (3, 4). Deep and beautiful proofs were found (5–7).

Classical hypercontractivity states $\|e^{-tH}\|_{L^2 \to L^2} \leq 1$, where $H$ is a simple harmonic oscillator Hamiltonian with unit angular frequency and $e^{2t} \geq q - 1 \geq 1$ (8–10). The classical Hausdorff–Young inequality is a consequence of $F = e^{itH/2}$ (5). Further inequalities can be found in many papers such as refs. 11–18, suggesting, in retrospect, a bridge from CFA to QFA.

1. QFA

A quantum Fourier transform $\mathcal{F}$ defines Fourier duality between quantum symmetries, which could be analytic, algebraic, geometric, topological, and categorical. The quantum symmetries could be finite or infinite, discrete or continuous, commutative or noncommutative. In certain contexts $\mathcal{F}$ can be defined pictorially—as in the picture language program (19). QFA is the study of structures involving $\mathcal{F}$.

It is possible to estimate various norms $\|\mathcal{F}\|_{L^p \to L^q}$ as transformations between noncommutative $L^p$ spaces, and results in refs. 20 and 21 represent early breakthroughs in the application and formulation of QFA. As QFA is more sophisticated than CFA, these subjects have differences as well as similarities; we explore them both.

Let us consider an example of similarities and differences. In CFA, the extremizers of the Hausdorff–Young inequality (and many others) are Gaussians. In QFA, on subfactors, the Hausdorff–Young also holds. The extremizers are bishifts of biprojections. A biprojection is a projection whose quantum Fourier transform is a multiple of a projection (22); so the behavior under Fourier transformation of the extremizers in QFA on subfactors is similar to those in CFA, while their algebraic properties differ.

In this paper, we give a unified view of QFA. We establish a “relative” inequality between pictures that yields an uncertainty principle for relative entropy. We propose a universal quantum inequality, namely Eq. 9, that unifies many other quantum inequalities. This is similar to the way the Brascamp–Lieb inequality unites Young’s inequality, Hölder inequality, and others, in CFA. Throughout the paper, we cite applications of QFA. Finally, in section 9, we state some general goals for the future and some open questions.

QFA reveals insight and intrinsic structure, as well as relations between fusion rings, fusion categories, and subfactors. We show how the “Schur product property” provides a powerful obstruction to distinguish mathematical objects. QFA also provides an approach to quantum entanglement, uncertainty relations, and other problems in quantum information. We are certain that QFA will lead to other advances in many different fields.

2. QFA on Fusion Rings

Let us start with fusion rings, as introduced by Lusztig (23); this is an interesting quantum symmetry beyond groups. See ref. 24 for further results and references. A fusion ring $\mathcal{A}$ is a ring that is free as a $\mathbb{Z}$-module, with a basis $\{x_1, x_2, \ldots, x_m\}$, $m \in \mathbb{N}$, with $x_1 = 1$, and such that

Significance

Classical Fourier analysis, discovered over 200 years ago, remains a cornerstone in understanding almost every field of pure mathematics. Its applications in physics range from classical electromagnetism to the formulation of quantum theory. It gives insights into chemistry, engineering, and information science, and it underlies the theory of communication. Quantum Fourier analysis extends this perspective. It yields insights and inequalities associated with uncertainty principles for quantum symmetries. In this paper, we introduce this mathematical subject, we show how it can solve some theoretical problems, and we give some applications to quantum physics with bounds on entropy and the analysis of quantum entanglement. We believe that quantum Fourier analysis, now in its infancy, will have significant future impact.


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Modern subfactor theory was initiated by Jones in 1983 by his index theorem asserting that \( \tau(x) = \delta_{+1} \), where \( \tau \) is a strictly positive trace, and therefore it defines an inner product. This gives the \( C^* \)-algebra \( \mathcal{B} \) by the Gelfand–Naimark–Segal (GNS) construction.

One can define a second (in this case abelian) \( C^* \)-algebra \( \mathcal{A} \) from \( \mathfrak{A} \) with multiplication \( \circ \), adjoint \# , given by another strictly positive linear functional \( d \) on \( \mathcal{A} \). Let

\[
\begin{align*}
[A1] & \quad x_0 \circ x_0 = \delta_{+1} d(x_0) = x_0, \\
[A2] & \quad x_0^\# = x_0.
\end{align*}
\]

Here, \( d(\cdot) \) is defined to be linear, by sending a basis element \( x_i \) to \( d(x_i) \), the operator norm of the fusion matrix \( M_{ij} \), with entries \( (M_{ij})_{+1} = N_{ij} \). This is the so-called Perron–Frobenius dimension of \( x_i \). The trace \( d(\tau) \) on the finite-dimensional \( C^* \)-algebra \( \mathcal{A} \) defines \( d \) on \( \mathcal{B} \) by \( \|a\|_{\mathcal{A}} = d(\|a\|_1^1) \) and \( \|b\|_{\mathcal{B}} = \tau(|\|b\|_1|) \). Then \( \mathcal{A} \) and \( \mathcal{B} \) are two \( C^* \)-algebras with the same basis. We use the classical notation \( \hat{x} : x \mapsto \hat{x} \) for the Fourier transform as the linear map from \( \mathcal{B} \) to \( \mathcal{A} \) defined by \( \hat{x}(x) = x \). The Fourier transform can be extended to a map from \( L^p(\mathcal{A}, d) \) to \( L^q(\mathcal{B}, \tau) \) for \( 1/p + 1/q = 1 \).

One has \( \tau(\hat{\hat{\delta}}(x)) \delta(y) = d(x \circ y) \).

We summarize several results in ref. 25 about QFA on fusion rings, including the quantum Schur product theorem (QSP), the quantum Hausdorff–Young inequality (QHY) with \( 1/p + 1/q = 1 \), the quantum Young inequality (QY) with \( 1/p + 1/q = 1 + 1/r \), and the basic quantum uncertainty principles (QUP)—defined in terms of the von Neumann entropy, \( H_\mathcal{A}(|x|^2) = -\log(\log|x\circ x|) \), and \( H_\mathcal{B}(|\hat{x}|^2) = -\log(\log|\hat{x}\circ \hat{x}|) \), and in terms of the support \( S_\mathcal{A}(x) = d(R(x)) \), \( S_B(x) = \tau(R(x)) \), where \( R(x) \) is the range projection.

**Theorem 2.1.** For nonzero \( x, y, z \in \mathcal{A} \),

- \([\text{QSP}]\) \( \hat{x}^{-1}(\hat{y}) \hat{z} \geq 0 \) whenever both \( x, y \geq 0 \).
- \([\text{QHY}]\) \( \|\hat{x}\|_{\mathcal{B}} \leq \|\hat{x}\|_{\mathcal{A}} \) for \( 1 < p < 2 \).
- \([\text{QY}]\) \( \|x \circ y\|_{\mathcal{A}} \leq \|x\|_{\mathcal{A}} \cdot \|y\|_{\mathcal{A}} \) for \( 1 \leq p, q, r < \infty \).
- \([\text{QUP-1}]\) \( H_\mathcal{A}(|x|^2) + H_\mathcal{B}(|\hat{x}|^2) + 2 \|x\|_{\mathcal{A}} \leq \log(\|x\|_{\mathcal{A}}) \geq 0 \).
- \([\text{QUP-2}]\) \( S_\mathcal{A}(x) = S_\mathcal{B}(x) \geq 1 \).

**Applications.** An important application of the Schur product theorem comes in the classification of abelian subfactor planar algebras (20). One can regard this as the fundamental theorem for abelian subfactors, extending the fundamental theorem for finite abelian groups. This was the first classification that requires a bound neither on the Jones index nor on the dimension.

**Theorem 3.2.** An irreducible subfactor planar algebra is abelian if and only if it is a free product of the Temperley–Lieb–Jones planar algebras and finite abelian groups.

Also one obtains a geometric proof of the (mathematical-physics version) of reflection-positivity condition.

**Theorem 3.3.** Reflection positivity holds for Hamiltonians on planar para algebras (theorem 7.1 in ref. 30) and on Levin–Wen models (theorem 3.2 in ref. 31).
The constant $\delta$ is the square root of the Jones index. The extremizers of these inequalities have nine different characterizations. In particular, the red line $1/p + 1/q = 1$, for $1/2 \leq 1/p \leq 1$ corresponds to the quantum Hausdorff–Young inequality. Moreover, all of the other quantum inequalities, such as quantum Young’s inequality, in Theorem 2.1 have been proved for subfactors of planar algebras in ref. [34].

4. QFA on Unitary Fusion Categories

The QFA on subfactors also applies for unitary fusion categories through the quantum double construction, see e.g., ref. 33. Let $C$ be a unitary fusion category and $I = \{X_1, X_2, \ldots, X_m\}$ be the set of simple objects. There is a Frobenius algebra $\gamma$ in $C \otimes C$ whose object is $\bigotimes_{i=1}^m X_i \otimes X_i$. Following the quantum double construction, we obtain an irreducible subfactor planar algebra, such that $\mathcal{A} = \mathcal{D}_{2,+} = \text{hom}_{C^\text{op}}(\gamma)$ and $\mathcal{B} = \mathcal{D}_{2,-} = \text{hom}_{\gamma \otimes \gamma}(\gamma \otimes \gamma)$. Applying QFA to $\mathcal{A}$ on this subfactor, we obtain inequalities on the Grothendieck ring of unitary fusion categories as stated in Theorem 2.1. Applying QFA to $\mathcal{B}$, we obtain inequalities on the dual of the Grothendieck ring, which turn out to be highly nontrivial.

Application: Analytic Obstructions. It is important to determine whether a fusion ring can be the Grothendieck ring of a unitary fusion category. QFA provides powerful analytic obstructions to the unitary categorification of fusion rings. The quantum inequalities in Theorem 2.1 holds on the dual of Grothendieck rings. However, they may not necessarily hold on the dual of fusion rings, thereby providing analytic obstructions of the unitary categorification of fusion rings. Even the Schur product property on the dual of a fusion ring, $0 \leq x \ast y$ if $0 \leq x, y \in \mathcal{B}$, gives by itself a surprisingly efficient analytic obstruction:

**Theorem 4.1** (25). If a fusion ring can be unitarily categorified, then the Schur product property holds on its dual.

There are 34 examples in the classification of simple integral fusion rings up to rank 8 and Frobenius–Perron dimension less than 3780. Four of them are group-like. Methods based on previously known analytic, algebraic, and number theoretic obstructions did not determine whether the remaining 30 could be unitarily categorified. As a consequence of the Schur-product obstruction, 28 out of the 30 have no unitary categorization, as shown in ref. 25.

**Example:** Let us recall one example from ref. 25 to illustrate this obstruction. Let $\mathfrak{X}$ be the rank-7 simple integral fusion ring with the following seven fusion matrices, equal to

![Fusion Matrices](image)

The eigenvalue table of these matrices (where $\zeta = 1$) is:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
5 & -1 & -\zeta - \zeta^6 & -\zeta^2 - \zeta^3 & -\zeta^2 - \zeta^4 & 0 & 0 \\
5 & -1 & -\zeta^2 - \zeta^3 & -\zeta^4 - \zeta^6 & -\zeta^3 & 0 & 0 \\
5 & -1 & -\zeta - \zeta^6 & -\zeta^6 & -\zeta^3 - \zeta^4 & -\zeta^2 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & -3 \\
7 & 1 & 0 & 0 & 0 & 0 & -1 \\
7 & 1 & 0 & 0 & 0 & 0 & -2 \\
\end{pmatrix}
\]

The first column is the Perron–Frobenius dimension of the seven simple objects. Take $X = x_1 + x_5 - 3x_6 + 2x_7$, then $X = X^* = X^2/15$.

The Schur product property on $\mathcal{B}$, equivalent to a dual version of Eq. 4, yields (with $x = y = z = X$) that

\[
d(\mathbf{XX}^*) \geq 0.
\]

However, it follows directly that Eq. 7 is false in this case, as

\[
\frac{1^3 + 0^3}{1} + \frac{0^3}{5} + \frac{0^3}{5} + \frac{0^3}{5} + \frac{1^3}{6} + \frac{(-3)^3}{7} + \frac{2^3}{7} = -\frac{65}{42} < 0.
\]

Therefore, the fusion ring $\mathfrak{X}$ cannot be unitarily categorified.

5. QFA on Locally Compact Quantum Groups

The previous results focus on finite quantum symmetry, such as fusion rings and finite-index subfactors. One might ask whether QFA can be established for infinite quantum symmetry. The answer is “yes”; there are results on infinite-dimensional Kac algebras and locally compact quantum groups. This relies on the theory of noncommutative $L^p$ spaces (e.g., ref. 34).

We recall the definition of the Fourier transform on locally compact quantum groups, of which the Fourier transform on Kac algebra is a special case (35). Let $G$ be a locally compact quantum group and $\varphi$ the left Haar weight. Suppose $W$ is the multiplicative unitary, $\phi$ is a normal semifinite faithful weight on the commutant of $L^\infty(G)$, and $\bar{d}$ is a normal semifinite faithful weight on the commutant of $L^\infty(\hat{G})$. Let $\frac{d}{\bar{d}}, \bar{d} = \frac{d}{2\pi}$ be the Connes’ spatial derivatives, and let $L^p(\varphi), L^p(\phi)$ be Hilsum’s space for any $1 \leq p < \infty$. The Fourier transform $\mathcal{F}_p : L^p(\phi) \to L^p(\hat{\phi})$, for $1 \leq p \leq 2$, and $1/p + 1/q = 1$, is defined by

\[
\mathcal{F}_p(x^{1/p}) = (\varphi \otimes \varphi)(W(x \otimes 1))d^{1/q}, \forall x \in \mathcal{L}_p
\]

Here, $\mathcal{T}_p \subset \mathcal{N}_p \cap \mathcal{N}_q^*$ is the space of elements analytic with respect to $\varphi$. Even the definition of the convolution on locally compact quantum groups is nontrivial.

The quantum inequalities in Theorem 2.1 on these infinite quantum symmetries have been partially studied in refs. 36–40. The quantum uncertainty principle $QU$--2 in Theorem 2.1 becomes a continuous family of inequalities on locally compact quantum groups (40).

6. Surface Algebras and A Universal Inequality

**Surface Algebras.** Many inequalities in CFA have not been axiomatized in a pictorial framework. Z.L. introduced surface algebras in ref. 41, formalizing the extension of planar algebras from two-dimensional (2D) to three-dimensional (3D) space,
outlined in ref. 42. Surface algebras are an extensive framework to capture additional pictorial features of Fourier analysis.

For any subfactor planar algebra, the actions of planar tangles can be further extended to the actions of surface tangles. (The arrow in planar diagrams corresponds to the $\delta$ sign in planar algebras. The clockwise/anticlockwise orientation of the arrow indicates the input/output disc in surface algebras.) One can represent Fourier transform, multiplication, and convolution as the action of the following surface tangles in the 3D space:

Using 3D pictures, one can consider the Fourier duality for surface tangles with multiple inputs and outputs. The 3D formalism has provided deep insights into quantum information, algebraic identities, and various other connections with physics.

One can consider a finite-dimensional Kac algebra $A$ as $A$ and its dual $\hat{A}$ as $B$, with a Fourier transform from $A$ to $\hat{A}$ defined analogously to section 5. The pair of Kac algebras $A$ and $\hat{A}$ can be understood as $A$ and $B$ for the surface algebra. The comultiplication is given by the picture:

The Hopf-axiom that the comultiplication is an algebraic homomorphism reduces to the string-genus relation of surface tangles given in equation 17 of ref. 42.

**A Universal Inequality.** In a subfactor planar/surface algebra $\mathcal{P}$, the Fourier transform, the multiplication, and the convolution can be realized by planar/surface tangles. In general, a surface tangle is a multilinear map on $\bigotimes_{a \in \mathcal{P}} \mathcal{P}_{a \pm}$. Now, we give a pictorial inequality in the quantum case, motivated by the classical Brascamp–Lieb inequality. We replace the dual of the linear map $B_j : \mathbb{R}^n \to \mathbb{R}^b$ by a surface tangle $T_j$ with $k_i$ input discs and $n$ output discs; moreover, the $n$-output discs are identical for different $j$:

$$\left\| \prod_{j=1}^{m} T_j(x_j) \right\|_1 \leq C \left\| x_j \right\|_{p_j},$$

and $C$ is the best constant.

This topological inequality includes the quantum Hausdorff–Young inequality, quantum Hölder inequality, and quantum version of Young’s inequality. The best constants of these three inequalities are achieved at biprojections.

For those familiar with the Quon language (42), we can consider the pictorial inequalities whose $T_j$s are surface tangles with braided charged strings. In particular, if all of the inputs and outputs are 2-boxes, corresponding to qudits, then $T_j$ can be any Clifford transformation on qudits. These Clifford transformations can be considered as a quantum analog of the dual of a linear transformation $B_j : (\mathbb{Z}_d)^n \to (\mathbb{Z}_d)^b$. Considering the action on density matrices, the $n$-qudit Clifford gates on Pauli matrices are symplectic transformations on $2n$-dimensional symplectic spaces over $\mathbb{Z}_d$.

### 7. Relative Inequalities, Entropy, and Uncertainty

Here, we present a relative, quantum, Hausdorff–Young inequality. This leads to a relative, quantum, entropic uncertainty principle.

Let $\mathcal{P}$ be an irreducible subfactor planar algebra with the Markov trace $tr$. Let $\varphi$ (resp. $\psi$) be a faithful state on $\mathcal{P}_{2,+}$ (resp. $\mathcal{P}_{2,-}$). Let $D_\varphi$ (resp. $D_\psi$) be the density operator of $\varphi$ (resp. $\psi$), namely, $\varphi(\cdot) = tr(D_\varphi \cdot)$. Now, we define a Fourier transform $\widehat{\delta}_{\varphi,\psi} : L^p(\mathcal{P}_{2,+}) \to L^q(\mathcal{P}_{2,-})$ for $1 \leq p \leq 2$, $q = p/(p-1)$ as

$$\widehat{\delta}_{\varphi,\psi}(x D_\varphi^{1/p}) = \delta(x D_\psi^{1/q}) D_\varphi^{1/q-1/2}. \quad [10]$$

This Fourier transform is represented pictorially as follows,

$$\widehat{\delta}_{\varphi,\psi} : \begin{array}{c|c|c} x D_\varphi & \widehat{\delta}_{\varphi,\psi} & \varphi D_\psi \end{array} \rightarrow \begin{array}{c|c|c} \varphi D_\psi & \delta(x D_\psi^{1/q}) D_\varphi^{1/q-1/2} & 0 \end{array}. \quad [11]$$

From Plancherel’s theorem for $\delta$, we infer Plancherel’s theorem for $\widehat{\delta}_{\varphi,\psi}$,

$$\left\| \widehat{\delta}_{\varphi,\psi}(x D_\varphi^{1/p}) \right\|_q \leq K_{\varphi,\psi} \left\| x D_\varphi^{1/p} \right\|_p. \quad [12]$$

**Theorem 7.1 (Relative, quantum Hausdorff–Young inequality).** Let $\mathcal{P}$ be an irreducible subfactor planar algebra and let $\varphi, \psi$ be faithful states on $\mathcal{P}_{2,\pm}$. Then for any $x \in \mathcal{P}_{2,\pm}$, $1 \leq p \leq 2$, and dual $2 \leq q = p/(p-1)$, we have

$$\left\| \delta_{\varphi,\psi}(x D_\varphi^{1/p}) \right\|_q \leq K_{\varphi,\psi} \left\| x D_\varphi^{1/p} \right\|_p. \quad [13]$$

Here, $K_{\varphi,\psi} = \delta^{-2/p} \left\| D_\varphi^{1/q-1/2} \right\|_1 \parallel \delta(D_\varphi^{1-1/p})_1. \quad [14]$

**Proof.** We give the elementary and insightful picture proof:

The first inequality is a consequence of the quantum Hölder inequality. The second inequality is a consequence of the quantum Hausdorff–Young inequality (theorem 4.8 in ref. 21). Here, the constant is $K = \sup_{\|s\|_1 = 1} K(y)$, with $\tilde{K}(y)$ equal to the following picture:
To obtain the first inequality, we use the quantum Young inequality (theorem 4.13 in ref. 21). To obtain the second inequality, we use the quantum Hölder inequality.

**Relative Entropy.** We formulate relative entropy (RE) and the corresponding relative entropic quantum (REQ) uncertainty principle. For two positive functionals \(\omega, \varphi\) on \(\mathcal{P}_{2,+}\), recall that the relative entropy (43) is

\[
S(\omega|\varphi) = \text{tr}(D_\omega(\log D_\omega - \log D_\varphi)).
\]

The **Relative Entropic Quantum Uncertainty Principle.** For a positive functional \(\omega\) on \(\mathcal{P}_{2,+}\), define \(\tilde{\omega}\) as the positive functional on \(\mathcal{P}_{2,-}\) given by the density matrix

\[
D_\omega = |\tilde{\omega}(D_{\tilde{\omega}}^{1/2})|^2.
\]

It follows that \(\tilde{\omega}(1) = \omega(1).\) If \(\omega\) is a state, so is \(\tilde{\omega}\).

**Theorem 7.2 (REQ Uncertainty Principle).** Let \(\mathcal{P}_*\) be an irreducible subfactor planar algebra and \(\varphi, \psi\) be faithful positive functionals on \(\mathcal{P}_{2,+}\). Then, for any state \(\omega\) on \(\mathcal{P}_{2,+}\),

\[
S(\omega||\varphi) + S(\tilde{\omega}||\psi) \leq \log \|D^{1/2}\|_\infty - \frac{1}{\delta^2} \log (\|D_\varphi\|_\infty) - 2 \log \delta.
\]

**Proof.** Note that \(\tilde{\omega}(D_{\tilde{\omega}}^{1/2}D_{\varphi}^{1/2}) = \tilde{\omega}(D_{\tilde{\omega}}^{1/2}D_{\varphi}^{1/2})\).

As \(\|AB\|_\infty = \|A\|_\infty \|B\|_\infty\), using Eq. 15, we infer that \(\tilde{\omega}(D_{\tilde{\omega}}^{1/2}D_{\varphi}^{1/2})\) and \(D_{\tilde{\omega}}^{1/2}D_{\varphi}^{1/2}\) have the same norms. Define the function \(f(p)\) as a picture, where \(q = p/(p - 1)\), and where \(K_{p,q}\) is defined in Eq. 14.

\[
f(p) = \begin{cases} 1 & \text{if } p = 2 \\ \frac{1}{2} & \text{if } p = 2 \\ 0 & \text{otherwise} \end{cases}
\]

The picture \(f(p)\) is negative for \(1 \leq p \leq 2\), by Theorem 7.1. Also \(f(2) = 0\) by Plancherel’s theorem, so the left derivative \(f’(2) > 0\). Then, Theorem 7.2 is a consequence of the expressions for the derivatives in the following lemma.

**Lemma 7.3.** For any positive functional \(\omega\) on \(\mathcal{P}_{2,+}\), we have

\[
\frac{d}{dp} \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix} = -\frac{1}{4} \log (\|D_\omega\|_\infty)
\]

\[
\frac{d}{dp} \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix} = -\frac{1}{4} \log (\|D_\omega\|_\infty)
\]

\[
\frac{d}{dp} \left( \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix} \right) = -\frac{1}{4} \log (\|D_\omega\|_\infty)
\]

8. QFA and Quantum Entanglement

Here, we use pictures as in refs. 30 and 44, which do not require shading. The Fourier transform of a multiple of the projection onto the zero-vector for the group \(\mathbb{Z}_d\), namely \(d^{1/2} |0\rangle \langle 0|\), is the identity.
projections $P$, $Q$ and constant $\lambda$, then there is a biprojection $B$, such that $\|x - B\| < \varepsilon$.

**Question 9.5.** Can one characterize the extremizers for the uncertainty principles on $n$-boxes, for $n \geq 3$?

**Block Renormalization Map and Quantum Central Limit Theorem.** The block map $B_\lambda$ is a composition of convolution and multiplication,

$$B_\lambda \left( \frac{1}{\|x\|_2^2} \right) = \delta_\lambda^2 \left( \frac{\lambda}{\|x\|_1} + \frac{(1 - \lambda)}{\|x\|_\infty} \right).$$

The limit points of the iteration of the block map are all biprojections for finite-index, irreducible subfactors (47). We regard this result as a quantum 2D central limit theorem.

**Conjecture 9.6.** For any $f \in L^\infty(R^n) \cap L^1(R^n) \cap L^2(R^n)$, $f$ converges either to 0 or to a Gaussian function, under the action of the iteration of the block map $2^n B_\lambda$.

**Conjecture 9.7.** For any $f \in L^\infty(R^n) \cap L^1(R^n) \cap L^2(R^n)$, the Hirschman–Beckner entropy decreases under the action of the block map $2^n B_\lambda$. The same question remains for finite cyclic groups.

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