Evolution in pecunia

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The paper models evolution in pecunia—in the realm of finance. Financial markets are explored as evolving biological systems. Diverse investment strategies compete for the market capital invested in long-lived dividend-paying assets. Some strategies survive and some become extinct. The basis of our paper is that dividends are not exogenous but increase with the wealth invested in an asset, as is the case in a production economy. This might create a positive feedback loop in which more investment in some asset leads to higher dividends which in turn lead to higher investments. Nevertheless, we are able to identify a unique evolutionary stable investment strategy. The problem is studied in a framework combining stochastic dynamics and evolutionary game theory. The model proposed employs only objectively observable market data, in contrast with traditional settings relying upon unobservable investors' characteristics (utilities and beliefs). Our method is analytical and based on mathematical reasoning. A numerical illustration of the main result is provided.

evolutionary finance | evolutionarily stable investment strategies | survival | stochastic dynamics | local stability

Despite bulls and bears adorning financial investors' desks and the financial press and traders' assurance that "it's a jungle out there," the emergence of evolutionary models in finance has been slow. However, as witnessed by the present special issue, the research on evolutionary finance (EF) is gaining traction.

Evolutionary ideas have a long history in the social sciences going back to Malthus, who played an inspirational role for Darwin [see, e.g., Hodgson (1)]. Veblen (2) coined the term "evolutionary economics" and started a systematic use of the evolutionary approach in the social sciences (3). Schumpeter (4) laid the groundwork for evolutionary economics in the 20th century. A crucial role in the creation of this branch of economics was played by the works of Alchian (5), Boulding (6), Downie (7), D. Friedman (8, 9), M. Friedman (10), Hodgson (1, 11), Penrose (12), Nelson (13), and Nelson and Winter (14).

Research in EF was started by the Santa Fe Institute, and the first time the term "evolutionary finance" appears is in one of its publications dating back to 1995 [LeBaron (15)]. In the seminal Santa Fe Institute working paper "Market Force, Ecology, and Evolution," Farmer (16) argues that financial market models can benefit from reasoning analogous to models of biological evolution. In particular, it would be useful to make investment strategies and not investors the actors in the model. This shift parallels biological models in which the interaction of species and not that of individual organisms are considered. Indeed, in financial market models, one can write market demand for assets as the wealth-weighted average of investors' demand. Alternatively, one can group all investors' wealth following the same strategy into one entity and write market demand as the wealth-weighted average of investment strategies. While this is a trivial operation mathematically, it shifts the focus away from the intentions behind the investment strategies (e.g., utility maximization subject to expectations) toward the actions taken in financial markets.

Many empirical studies [for example, the well-known Fama and French factor models (17, 18)] have shown that a few strategies are sufficient to understand the dynamics of thousands of assets. Equating aggregate demand with the supply of assets, asset prices are then the wealth-weighted average of a few investment strategies. Thus, in order to understand asset returns, which are the ratios of next period prices (plus eventual dividends) to current period prices, one needs to understand the evolution of wealth behind investment strategies. As Farmer (16) notes, this evolution of wealth models the market selection process acting on the financial species, i.e., the investment strategies. While the market selection force reduces the variety of species in the financial markets, Farmer (16) also points out that there is a countervailing force that innovates new strategies. In biological evolution this is mostly done by sexual reproduction along which genes are recombined. In financial evolution, there are other methods of innovation including rational and behavioral aspects, e.g., backtesting and forward performance testing (often akin to adaptive heuristics in game theory [cf. Hart (19)]).

Achievements of EF

What has this new paradigm for finance achieved so far? On one hand, it has improved our understanding of the dynamics of asset prices since many stylized facts, such as, for instance, excess volatility, can be explained by the endogenous dynamics of wealth. Excess volatility was first pointed out by Shiller (20) who showed that the prices of the Standard and Poor's (S&P) 500 index are more volatile than the fundamental values computed with models of expected utility maximization given rational expectations. Boswijk et al. (21) showed that a simple EF model can explain the excess volatility of the S&P 500.

Significance

We use an analogy between financial trading strategies and biological species and characterize an evolutionary stable strategy in an ecology with fix-mix strategies. The basis of our paper is that dividends are not exogenous but increase with the wealth invested in an asset, as is the case in a production economy. While this might create positive feedback loops, we show that the dynamical system has a unique evolutionary stable investment strategy that characterizes a locally stable equilibrium state. Our result is of high significance for any market economy since it shows that the dynamic interaction of investment strategies tends to produce stable prices. Mispricing therefore occurs only in extraordinary times when the market becomes highly dislocated due to exogenous events.

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Other stylized facts that are hard to reconcile with utility maximization include stochastic volatility, autocorrelation, and heavy tails in the return distribution of asset prices [cf. Cont (22) for a more exhaustive list]. For comprehensive treatments of the achievements of EF in asset pricing we refer to LeBaron (23) and Hommes (24).

On the other hand, EF also contributes to portfolio theory, which is not descriptive but normative. Portfolio theory asks how to invest. The traditional answer [see, for example, Markowitz (25)] is that one should maximize an objective function given the return expectation one has. In this view, returns are taken as exogenous. However, modeling the financial market via a few investment strategies, the impact of the strategies (not the individual investors) on the market is obvious, and a game theoretic approach would be more suitable. One should select a strategy that performs well in competition with the other strategies. Performing well in evolutionary models means at least to stay alive. Thus, in evolutionary portfolio theory there is a focus on so-called survival strategies. Applying this idea to the evolution of relative wealth, survival requires that no other strategy achieves a higher growth rate of wealth. Of course, this criterion has always been criticized by adherents of utility maximization [see, e.g., Samuelson (26)], but as Sciubba (ref. 27, p. 125) put it eloquently, a survival strategy "might not make you happy, but will definitely keep you alive."

One might suspect that the existence and the characterization of survival strategies depend on the exogenous stochastic process and on the market ecology, i.e., the set of investment strategies competing for wealth. This is indeed the case when one limits the pool of strategies. However, since there is always a potential for innovation, it would be risky to do so. Indeed the most general result on survival strategies that was achieved so far [Evstigneev et al. (28)] shows the existence of a survival strategy for any ecology of investment strategies and any dividend processes. The survival strategy can be characterized as being a well-diversified fundamental strategy, which is contrarian. As such, it might explain the great success of value investing in equity markets [cf. Gergaud and Ziemba (29)]. The survival strategy found in Evstigneev et al. (28) is a so-called basic strategy because it only conditions on the exogenous stochastic process of dividends. In addition, within the set of basic strategies, it is unique. In general, however, there might exist other (nonbasic) strategies achieving the same growth rate of wealth as the survival strategy.

How does this result square with other results in EF? The result of Evstigneev et al. (28) is purely analytical, while most other related results in EF are based on simulations. Thus, the conditions under which the result of Evstigneev et al. (28) is obtained are clearly understood.

Furthermore, most other results in the literature are based on a limited set of strategies—not allowing all innovations. Limiting the market ecology has been a successful strategy to better understand asset prices. For example, the paper of Scholl et al. (30) in this special issue limits the ecology to a fundamental, a momentum, and a noise trader strategy and is able to explain many interesting stylized facts of asset prices. Surely, models explaining stylized facts of asset pricing get stronger the simpler they are. However, such a limitation is potentially dangerous when one wants to draw general conclusions for portfolio theory. A strategy that is best in a restricted ecology might suffer severe losses when a new strategy from outside the current ecology emerges. A similar remark applies to the famous Brock and Hommes model (24), which is also based on a similar set of three types of strategies but enriches the evolution of wealth by allowing investors to switch between the three strategies. As a result, much richer asset price dynamics may be achieved. However, as Hens and Schenk-Hoppé (31) have shown, introducing a strategy that stolidly follows the fundamental strategy of Evstigneev et al. (28) would drive out all other strategies of the Brock and Hommes model.

Finally, results in EF depend on the market microstructure. In the famous Santa Fe model (cf. ref. 23), strategies are generated by genetic algorithms, and markets are cleared by a market maker. As was shown in ref. 32, also using genetic algorithms, the survival strategy of Evstigneev et al. (28) will evolve when one uses a batch auction as in Evstigneev et al. (28)*.

A survey describing the state of the art in the field by 2016 and outlining a program for further research is given in ref. 35. An elementary textbook treatment of the subject can be found in Evstigneev et al. (ref. 36, chap. 20). For a most recent review of studies related to EF, see Holtfort (37). A novel line of research in EF considering models with endogenous asset supply was initiated in Amir et al. (38).

Contribution of This Paper to EF

The present paper draws on previous work by Evstigneev et al. (39), where a prototype of the model studied here was developed and some versions of the results of the present paper were obtained. The main basis of the modeling framework considered here lies in the fact that the dividends paid by the assets depend not only on exogenous random factors but also on the fraction of total market wealth invested in each particular asset. This is an important extension of EF models since in reality, dividends do not fall from trees [as in the famous Lucas (40) model] but are produced by firms that use capital as one of their inputs. The more wealth is invested in the outstanding stocks of a company, the easier it is to raise new capital and thus to produce more dividends.

This claim follows from a long tradition of capital-based asset pricing models. First, Tobin (41) claimed and Tobin and Brainard (42) gave evidence that firms increase their capital stock if their market value increases above the value of their capital in place, i.e., above their book value. This is the famous q-theory of investments according to which the ratio of book to market is essential for investments. Li et al. (43) estimate the production function that is implicit in q-theory as a concave power function determining profits from the amount of capital employed by the firm.

Finally, as Lintner (44) first showed, dividends are a fixed proportion of profits, so that our assumption that dividends depend on the market capitalization of the firm has a good foundation in finance. Indeed, Fig. 1 shows market capitalization and dividend data of three firms that have dividend payouts in each year of the sample 1981 to 2019.† We fit a dividend production function of the form $(c_1 b)^{c_2}$, where b is the firm's capital and c_1, c_2 are firm-specific and estimated. We find that the average relation between the market capitalization of dividend-paying firms and their total cash dividends is given by such a concave function. Of course, these dividend functions differ across firms. A limitation of our current model is that it does not capture firms that do not pay dividends or make any other disbursements to shareholders.

As we illustrate below, the average relation between the market capitalization of dividend-paying firms and their total cash dividends is given by a simple increasing and concave function. Of course, these dividend functions differ across firms. The limitation of our current model is that it does not capture firms that do not pay dividends or make any other disbursements to shareholders.

*That the asset price dynamics of EF models depends on the market microstructure is shown in Bottazzi et al. (33) and Anufriev and Panchenko (34). The point made in Lensberg and Schenk-Hoppé (32) is to show that also the outcome of the market selection depends on the market microstructure.

†The data are available in Dataset S1.

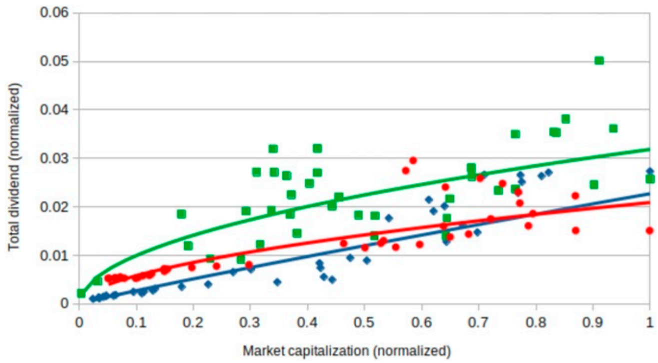

Fig. 1. Relation between market capitalization and total dividend payouts. The solid lines are the result of a linear regression of logarithms of both quantities and estimated separately for each firm. To compare the shape of the dividend production function across companies, each firm's market capitalization and dividend payout is divided by the firm's maximum market capitalization in the sample. General Motors is shown in green, Exxon is in red, and Procter & Gamble is in blue. The fits, measured as R^2, are 0.73, 0.83, and 0.92, respectively.

We present a rigorous EF stock market model with endogenous dividends[‡]. This important extension of the EF model comes at a cost. For this paper we limit the attention to so-called fix-mix strategies, which hold the investment proportions constant over time. We note that the class of fix-mix, or constant proportions, strategies we consider in this work is quite common in financial theory and practice; see, e.g., Perold and Sharpe (46), Mulvey and Ziemba (47), and Browne (48).[§] Moreover, recent empirical evidence by DeMiguel et al. (49) has shown that even the simplest fix-mix strategy that invests the same fraction in all assets is at least as good as sophisticated mean-variance optimization strategies. Thus, from the practical and from the theoretical standpoints, this class of strategies provides a convenient laboratory for the analysis of questions we are interested in. It makes it possible to formalize in a clear and compact way the concept of a type (genetic code) of an investment strategy, which determines the evolutionary performance of its portfolio rule in the long run. From the practical standpoint, fix-mix strategies are of importance since under certain general conditions they might lead to endogenous growth of wealth—volatility-induced growth [Dempster et al. (50)]. Finally, it should be noted that in models with independent and identically distributed (i.i.d.) random factors, fix-mix strategies typically outperform all of the others (51), and we conjecture that this is the case for the model at hand, although a proof of this conjecture is not available at this point.[¶]

The strategies determine the ecology of the market and its random dynamics over time. In the evolutionary perspective, the outcome of survival or extinction of investment strategies is governed by the long-run behavior of the relative wealth of the strategies, which in turn depends on the combination of the strategies in the ecology. A strategy is said to survive if it generates with probability 1 a strictly positive share of market wealth, bounded away from 0, over an infinite time horizon, irrespective of the set of investment strategies in the ecology. It is said to become extinct if the share of market wealth corresponding to it tends to 0.

An investment strategy, λ^* is called evolutionarily stable if the following condition holds. Suppose the ecology consists of $N-1$ strategies $2, 3, \ldots, N$ (nonmutants), a new strategy 1 (mutant) enters the existing ecology, and moreover, the initial share of wealth of this new strategy is small enough. Then the new strategy 1 will be driven out of the market by the other strategies in the long run: its market share will tend to 0 with probability 1 as time goes to infinity. This definition combines ideas from two fundamental solution concepts of evolutionary game theory proposed by Maynard Smith and Price (52) and Schaffer (53). We provide an effective construction of the evolutionarily stable strategy λ^* and trace its links to the famous Kelly portfolio rule of betting your beliefs [Kelly (54), Breiman (55), Thorp (56), Algoet and Cover (57), and Hakansson and Ziemba (58)].

Our main result—*Theorem* stated in the next section—demonstrates the existence and uniqueness of an evolutionary stable strategy (ESS) for our model. This result makes an important contribution to the asset pricing as well as to the portfolio theory aspect of EF. Our result recommends to investors to structure their portfolio based on fundamentals such as dividends. Moreover, the portfolio should be completely diversified and needs to be rebalanced over time, i.e., the investment proportions need to be restored after deviations resulting from price changes. If these rules are followed by all investors, then any other investment strategy will lose wealth relative to this fundamental strategy. If the market were governed by another strategy, then this strategy could survive since there exist better strategies that will gain against the incumbent strategy even when initially the entrant strategy has little wealth. Thus, in order to survive, it is necessary to follow the strategy we identified in *Theorem*.

Theorem gives support to the discounted cash flow rule, which is the classical asset pricing rule in traditional finance models with utility maximization given rational expectations. The price of any equity should be equal to the discounted sum of its future dividends[#]. However, there are important differences. First, *Theorem* shows that in order to survive, one needs to discount the future relative dividends. Second, observing these prices as the market outcome is more likely since this is the unique ESS, but this is not guaranteed, because global stability is unresolved. Finally, note that without rational expectations one would have to learn the process determining future dividends. A natural approach would be to estimate it from the history of dividends one has observed. As our model shows, this might, however, be misleading since actual dividends depend on the wealth invested in the assets. By the interaction of the heterogeneous strategies in the market we would expect to see quite complicated trajectories of realized dividends. Nevertheless, the ESS does not depend on the ecology of the market, neither when strategies are still competing with each other nor when the ESS is established.

The intuition for our main result, the identification of an evolutionarily stable investment strategy and its characterization, is as follows. As the capital invested in a particular investment strategy increases, the assets that are overweighted (relative to the market portfolio) become more expensive, lowering their returns. Likewise, assets that are underweighted become cheaper and see their returns increase. Both forces are to the disadvantage of the investment strategy at hand (and to the benefit of other investment strategies that have opposing weights). An

[‡]A similar feedback effect has been studied by Cherkashin et al. (45) in a much simpler setting. They analyze a model with short-lived assets—one for each state of the world - in which the probability of occurrence of a state of the world depends on the amount invested in the asset paying off in that state.

[§]In fact, such strategies are routinely solicited and used by pension and investment funds, such as Teachers Insurance and Annuity Association of America and College Retirement Equities Fund and Vanguard.

[¶]Numerical simulations of the model are described in *Illustration*.

[#]The traditional argument goes as follows: the price of equity today should be equal to the discounted payoffs the equity holding entitles to next period, i.e., equal to the resale value and the dividends being paid. Iterating this argument forward, at any point in time the price is then equal to the discounted sum of all future dividends.

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evolutionarily stable investment strategy must, therefore, with increasing capital, move prices into a direction that does not offer such an advantage to other strategies. As it turns out, a fundamental value in relative terms provides these conditions. Thinking in terms of growth rates in random dynamical systems, an evolutionarily stable investment strategy must imply asset returns such that no other investment strategy can have a positive growth rate.

The structure of the remainder of the paper is as follows. *Model and Results* describes the model and states the main result (*Theorem*)—a rigorous proof of the results can be found in *SI Appendix*, *Methods*. In *Illustration*, we provide an intuitive explanation and illustrate the formal arguments by a simulation. We conclude with *Discussion*.

Model and Results

We consider a market where $K \geq 2$ assets are traded at moments of time $t = 0, 1, \ldots$. The supply of each asset $k = 1, \ldots, K$ is constant (independent of time) and is denoted by V_k. There are $N \geq 2$ investment strategies interacting in the market. An investment strategy (portfolio rule) is represented by a nonnegative vector $\lambda^i = (\lambda^i_1, \ldots, \lambda^i_K)$ with components adding up to 1; i.e., it lies in the unit simplex Δ^K.

The market is influenced by random factors modeled in terms of a sequence of independent identically distributed elements $s_1, s_2, \ldots$ in a measurable space S. The random element s_t is interpreted as the state of the world at time/date t. The wealth of investment strategies $i = 1, 2, \ldots, N$ at date $t \geq 1$ is denoted by $w^i_t = w^i_t(s^t)$, where $s^t := (s_1, \ldots, s_t)$ stands for the history of states of the world up to date t. Initial endowments $w^i_0 > 0$ of all of the investment strategies at date 0 are given.

An investment strategy i at each time t allocates wealth w^i_t across assets $k = 1, \ldots, K$ in constant (independent of time and random factors) proportions λ^i_k.

Given the set of investment strategies λ^i, $i = 1, \ldots, N$, the total amount allocated for purchasing asset k at time t is expressed as

$$\langle \lambda_k, w_t \rangle := \sum_{i=1}^{N} \lambda^i_k w^i_t, \ \lambda_k := (\lambda^1_k, \ldots, \lambda^N_k), \ w_t := (w^1_t, \ldots, w^N_t). \qquad [1]$$

At each time $t = 1, 2, \ldots$, assets $k = 1, \ldots, K$ pay dividends

$$d_{t,k} = d_k(s_t, W_{t-1,k}) \geq 0 \qquad [2]$$

depending on the fraction

$$W_{t-1,k} := \frac{\langle \lambda_k, w_{t-1} \rangle}{\sum_{j=1}^{K} \langle \lambda_j, w_{t-1} \rangle} \qquad [3]$$

of total market wealth

$$W_{t-1} := \sum_{j=1}^{K} \langle \lambda_j, w_{t-1} \rangle = \sum_{i=1}^{I} w^i_{t-1} \qquad [4]$$

allocated to asset k. The functions $d_{t,k}(s, b)$, $b \in [0, 1]$, are assumed to be jointly measurable with respect to their arguments and satisfy

$$\sum_{k=1}^{K} d_{t,k} > 0. \qquad [5]$$

We denote by $p_t = p_t(s^t) \in \mathbb{R}^K_+$ the vector of market prices of the assets. For each $k = 1, \ldots, K$, the coordinate $p_{t,k}$ of the vector $p_t = (p_{t,1}, \ldots, p_{t,K})$ stands for the price of one unit of asset k at date t. Below we describe how these prices are formed in equilibrium over each time period. A portfolio of investment strategy i at date $t = 0, 1, \ldots$ is specified by a vector $x^i_t = (x^i_{t,1}, \ldots, x^i_{t,K}) \in \mathbb{R}^K_+$ where $x^i_{t,k}$ is the amount (the number of units) of asset k in the portfolio x^i_t. The scalar product $\langle p_t, x^i_t \rangle = \sum_{k=1}^{K} p_{t,k} x^i_{t,k}$ expresses the value of the investment strategy i's portfolio x^i_t at date t in terms of the prices $p_{t,k}$. The portfolio vector x^i_t depends on the history s^t of states of the world: $x^i_t = x^i_t(s^t)$. This vector function of s^t, as well as all of the other functions of s^t we deal with, is measurable. To alleviate notation, we will often omit s^t in what follows.

At date $t = 0$ the budgets are given by their (nonrandom) initial endowments $w^i_0 > 0$. Investment strategy i's budget/wealth at date $t \geq 1$ is

$$w^i_t = \langle d_t + p_t, x^i_{t-1} \rangle = \sum_{k=1}^{K} (d_{t,k} + p_{t,k}) x^i_{t-1,k}, \qquad [6]$$

where

$$d_t := (d_{t,1}, \ldots, d_{t,K}), \ d_{t,k} = d_k(s_t, W_{t-1,k}), \ k = 1, \ldots, K. \qquad [7]$$

The budget consists of two components: the dividends $\langle d_t, x^i_{t-1} \rangle$ paid by yesterday's portfolio x^i_{t-1} and the market value $\langle p_t, x^i_{t-1} \rangle$ of x^i_{t-1} expressed in terms of today's prices p_t. If investment strategy i allocates the fraction λ^i_k of wealth w^i_t to asset k, then the number of units of asset k that can be purchased for this amount is

$$x^i_{t,k} = \rho \frac{\lambda^i_k w^i_t}{p_{t,k}}, \qquad [8]$$

where $1 - \rho \in (0, 1)$ is the transaction cost factor. Thus, by employing the portfolio rule $\lambda^i = (\lambda^i_1, \ldots, \lambda^i_K)$, a portfolio is constructed whose positions are specified by Eq. **8**.

Suppose that strategies $\lambda^i = (\lambda^i_1, \ldots, \lambda^i_K) \in \Delta^K$ have been selected. Assume that the market is always in equilibrium: for all $t = 0, 1, \ldots$ and $k = 1, \ldots, K$, total asset supply is equal to total asset demand

$$V_k = \sum_{i=1}^{N} x^i_{t,k}, \qquad [9]$$

i.e.,

$$V_k = \rho \sum_{i=1}^{N} \frac{\lambda^i_k w^i_t}{p_{t,k}} \qquad [10]$$

(Eq. **8**). Then we get

$$p_{t,k} = \frac{\rho}{V_k} \sum_{i=1}^{N} \lambda^i_k w^i_t. \qquad [11]$$

By combining (**11**) and (**6**), we obtain a system of equations that determines the equilibrium (market clearing) prices

$$p_{t,k} = \frac{\rho}{V_k} \sum_{i=1}^{N} \lambda^i_k \langle d_t + p_t, x^i_{t-1} \rangle, \ k = 1, \ldots, K. \qquad [12]$$

It can be shown that a nonnegative vector p_t satisfying these equations exists and is unique (for any s^t, $d_t \geq 0$ and any feasible x^i_{t-1} and λ^i_t) (*SI Appendix*, Section 2, Proposition 1).

Given a strategy profile $(\lambda^1, \ldots, \lambda^N)$ and initial endowments $w^1_0, \ldots, w^N_0$, we can generate a path of the system by using

Eqs. **6–12**. Assume that all quantities are well defined (sufficient conditions are provided below), then the solution of Eq. **6**, which is linear in wealth, gives the explicit random dynamics

$$w_t = [Id - \rho\Theta_{t-1}\Lambda]^{-1}\Theta_{t-1}d_{t-1}, \quad \textbf{[13]}$$

$$d_t = \left(d_k\left(s_t, \langle\lambda_k, w_{t-1}\rangle / \sum_{j=1}^{K}\langle\lambda_j, w_{t-1}\rangle\right)\right)_{k=1}^{K}, \quad \textbf{[14]}$$

where Id is the $N \times N$ identity matrix, $\Theta_{t-1} = (x_{t-1,k}^i)_k^i$ is the matrix of portfolios, and Λ is the matrix of all investment strategies.

The above dynamics makes sense only if $p_{t,k} > 0$ or, equivalently, if the aggregate demand for each asset (under the equilibrium prices) is strictly positive. Those strategy profiles which guarantee that the recursive procedure described above leads at each step to strictly positive equilibrium prices will be called admissible.

We give a simple sufficient condition for a strategy profile to be admissible. It will hold for all of the strategy profiles we shall deal with in the present paper, and in this sense it does not restrict generality. Suppose that some investment strategy, say $i = 1$, uses a portfolio rule that always prescribes to invest into all of the assets in strictly positive proportions λ_k^1. Then any strategy profile containing this portfolio rule is admissible (see ref. 59, p. 167).

Let $(\lambda^1, \ldots, \lambda^N)$ be an admissible strategy profile. Consider the path of the asset market generated by this strategy profile and the given initial endowments $w_0^i > 0$, $i = 1, \ldots, N$. We are primarily interested in the long-run behavior of the relative wealth or the market shares $r_t^i := w_t^i / W_t$ of the investment strategies, where $W_t = \sum_{i=1}^N w_t^i$ is the total market wealth. The main concept we analyze in this paper is that of an ESS.

Definition: A portfolio rule λ^* is called evolutionary stable if it possesses the following property. Suppose there are two investment strategies, $\lambda^2 = \lambda^*$ and $\lambda^1 = \lambda \neq \lambda^*$. Furthermore, suppose that the initial market share r_0^1 of investment strategy 1 is small enough: $r_0^1 < \delta$, where $\delta > 0$ is some random variable. Then the market share r_t^1 of investment strategy 1 will tend to 0 almost surely; i.e., investment strategy 1 will be driven out of the market by the other investment strategy λ^* with probability 1.

The above definition of an ESS combines two fundamental concepts of evolutionary game theory: the classical definition of an ESS for continuous populations by Maynard Smith and Price (52) and its version for discrete populations proposed by Schaffer (53). The analogy with the former lies in the fact that the initial relative wealth of the mutant (λ-investment strategy) is assumed to be small enough; under this assumption, the λ-investment strategy cannot survive in competition with an incumbent (λ^*-investment strategy). A parallel with the latter is in the assumption that there is only one mutant type represented by the λ-investment strategy. Relative wealth is the counterpart of the relative mass of a continuous population of mutants or nonmutants in a biological context. A fundamental distinction between the notion introduced and the classical ones is that in the present EF setting we are dealing with properties holding with probability 1, while the classical biological notions of evolutionary stability are concerned with frequencies, probability distributions, and properties holding on average.

To formulate the main result of this work (*Theorem* below) we introduce some assumptions and notation. Put $D_k(s, b) := V_k d_k(s, b)$. This function represents the total amount of dividends paid by all of the assets k available in the market.

Assumption 1. *For each s and k the function $D_k(s, b)$ ($b \in [0, 1]$) is strictly positive, differentiable, strictly monotone increasing, and concave in b.*

Assumption 2. *For any $\lambda = (\lambda_1, \ldots, \lambda_K) \in \Delta^K$, the functions $D_k(s, \lambda_k)$ are linearly independent; i.e., if for some constants $a_1, \ldots, a_K$ the equality $\sum_{k=1}^K a_k D_k(s, \lambda_k) = 0$ holds for all s, then $a_1 = \ldots = a_K = 0$.*

Assumption 3. *There exist constants $D'_{\max} > 0$ and $D_{\min} > 0$ such that*

$$D'_{\max} < D_{\min} \quad \textbf{[15]}$$

and for all s, b, and k we have

$$D_k(s, b) \geq D_{\min}, \; D'_k(s, b) \leq D'_{\max},$$

where $D'_k(s, b)$ stands for the derivative of the function $D_k(s, b)$ with respect to b.

Assumption 1 contains standard regularity conditions on the functions $D_k(s, b)$, which are typical assumptions on a production function. *Assumption 2* means the absence of redundant assets: one cannot construct a synthetic asset, a portfolio with fixed weights consisting of assets $j \neq k$, that yields the same dividends as any given asset k. *Assumption 3* says that although the growth of the total investment in an asset k leads to a growth in the dividend paid by that asset, this growth is moderate: its rate $D'_k(s, b) = V_k d'_k(s, b)$ cannot exceed the constant specified in Eq. **15**. Such an assumption is natural when, in addition to capital, a second production factor (e.g., labor) is essential. Fig. 1 illustrates that a function $D(s, b) = (1 + s) \cdot (c_1 b)^{c_2}$ (c_1, c_2 are estimated, and s is a random variable which is determined by the ratio of actual and average dividend, given current market capital) makes sense empirically and can satisfy these assumptions.

Theorem. *There exists a unique solution $\lambda^* = (\lambda_1^*, \ldots, \lambda_K^*) \in \Delta^K$ to the system of equations*

$$E\frac{D_k(s, \lambda_k^*)}{\sum_{m=1}^K D_m(s, \lambda_m^*)} = \lambda_k^*, \; k = 1, 2, \ldots, K. \quad \textbf{[16]}$$

We have $\lambda_k^* > 0$, $k = 1, \ldots, K$. The portfolio rule represented by the vector λ^* is evolutionary stable.

In Eq. **16**, s is a random element in the space S having the same distribution as s_t ($t = 1, 2, \ldots$). The symbol E stands for the expectation with respect to this distribution. The meaning of Eq. **16** is as follows. It says that the relative dividends

$$R_k^*(s) := \frac{D_k(s, \lambda_k^*)}{\sum_{m=1}^K D_m(s, \lambda_m^*)} = \frac{V_k d_k(s, \lambda_k^*)}{\sum_{m=1}^K V_m d_m(s, \lambda_m^*)}$$

corresponding to the allocation of wealth across assets in the proportions $\lambda_1^*, \ldots, \lambda_K^*$ prescribed by the evolutionary stable portfolio rule λ^* coincide on average with these proportions.

If the functions $D_k(s, b)$ do not depend on b, Eq. **16** boils down to

$$\lambda_k^* = ER_k^*, \; k = 1, 2, \ldots, K.$$

In this case, λ^* reduces to the prescription to invest in accordance with the expected relative dividends. This is the classical Kelly portfolio rule—betting your beliefs [Kelly (54)][‖]. In EF models with exogenous dividends, stronger (global) versions of results of this kind were obtained in refs. 51, 59.

[‖] In the classical capital growth theory with exogenous asset returns [Kelly (54), Breiman (55), and Algoet and Cover (57)], the portfolio rule of betting your beliefs is obtained as a result of the maximization of the expected logarithm of the portfolio return. In our game-theoretic setting, where the performance of a strategy depends not only on the strategy itself but on the whole strategy profile, the evolutionary stable portfolio rule cannot be obtained as a solution to a single-agent optimization problem with a logarithmic, or any other, objective function.

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The intuition behind the main result can now be made more explicit using the random dynamics of relative wealth (see also *SI Appendix*, Proposition 2):

$$r_{t+1}^{i}=\sum_{k=1}^{K}[\rho\langle\lambda_k, r_{t+1}\rangle+(1-\rho)R_{t+1,k}]\frac{\lambda_k^i r_t^i}{\langle\lambda_k, r_t\rangle}, \qquad [17]$$

$i=1,\ldots,N,\ t\geq 0$.

Assume a fix-mix investment strategy β has all of the market wealth, then the return of asset k is

$$\left[\rho\beta_k+(1-\rho)\frac{D_k(s,\beta_k)}{\sum_{m=1}^{K}D_m(s,\beta_m)}\right]/\beta_k=:\mu_k(s,\beta)/\beta_k.$$

Treating these returns as exogenous, an investment strategy λ will see its wealth evolve as

$$w_{t+1}=\sum_{k=1}^{K}\frac{\mu_k(s,\beta)}{\beta_k}\lambda_k w_t,$$

and its growth rate $E\ln(w_{t+1}/w_t)$ as a function of λ is

$$g(\lambda):=E\ln\left(\sum_{k=1}^{K}\mu_k(s,\beta)\frac{\lambda_k}{\beta_k}\right). \qquad [18]$$

Assumption 2 implies that it is strictly concave, and $g(\beta)=0$ holds. Its total differential

$$dg(\beta)=\sum_{k=1}^{K}\frac{\partial g}{\partial\lambda_k}(\beta)d\lambda_k=\sum_{k=1}^{K}E[\mu_k(s,\beta)/\beta_k]d\lambda_k$$

is 0 for any λ if and only if $E[\mu_k(s,\beta)/\beta_k]=const$ (otherwise, because $\sum_k d\lambda_k=0$, there is a strategy with a strictly positive growth rate). This condition holds only if $\beta_k=E\mu_k(s,\beta)$ for all k. *Theorem* shows that only λ^* has this property. Evolutionary stability, however, is more demanding to show as one cannot assume that returns are exogenous, but one has to deal with the actual dynamics.

Moreover, Eq. **18** reveals that complete diversification, i.e., $\beta_k>0, k=1,\ldots,K$, is a necessary condition for an ESS. The characterization of the unique ESS in *Theorem* shows how an ESS needs to diversify. Note that from an evolutionary perspective diversification does not serve the purpose of achieving a high-risk adjusted return but to keep the growth rate of incumbent strategies low. One might call this “spiteful diversification.”

Illustration

A numerical example with time-dependent investment strategies is provided to illustrate 1) the capability of λ^* and 2) the pitfall of not including such a fundamental strategy in agent-based models of financial markets.

There are two assets in supply $V_k=k$ and dividends $d_k(s_t, W_{t-1,k})=1+s_t W_{t-1,k}$, $k=1,2$. The total amount of dividends paid by the asset k in period t is $D_k(s_t, W_{t-1,k})=V_k d_k(s_t, W_{t-1,k})$. The process s_t is i.i.d. and log-normal with parameters $(1,1)$.

There are three investment strategies. First, the ESS $\lambda^1=\lambda^*=(0.2,0.8)$, which is fixed over time. Second, λ_t^2 is a history-dependent, trend chaser (momentum) strategy. Denote by $R_{t-1,k}$ asset k's realized return from period $t-2$ to $t-1$ and by $\overline{R}_{t-1}$ its average over $k=1,2$. Then $\lambda_{t,1}^2:=\arctan(R_{t-1,1}-\overline{R}_{t-1})/\pi+0.5$ and $\lambda_{t,2}^2:=\arctan(R_{t-1,2}-\overline{R}_{t-1})/\pi+0.5=1-\lambda_{t,1}^2$. Since there are no previous returns in the initial period, the strategy is chosen randomly. Third, a noise trader strategy varies from period to period. In each period this strategy is determined by randomly drawing $\lambda_{t,1}^3$ uniformly from $[0,1]$ and setting $\lambda_{t,2}^3=1-\lambda_{t,1}^3$.

The simulation is carried out as follows. Strategies start with different initial wealth shares; λ^*'s initial share is $r_0^1\geq 0.9$. In

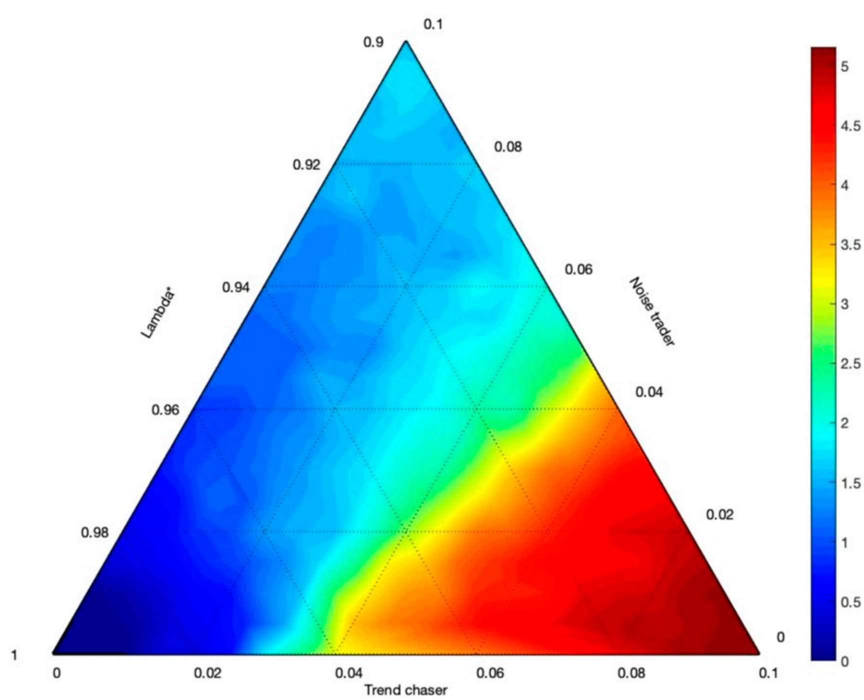

Fig. 2. Number of time periods until the relative price of the two assets is within 2.5% of the benchmark after the equilibrium is disturbed by the introduction of the trend chaser and the noise trader with total wealth below 10% of market capital. The number of time periods is the expected value calculated as the average over several runs and presented as natural logarithm to show the structure.

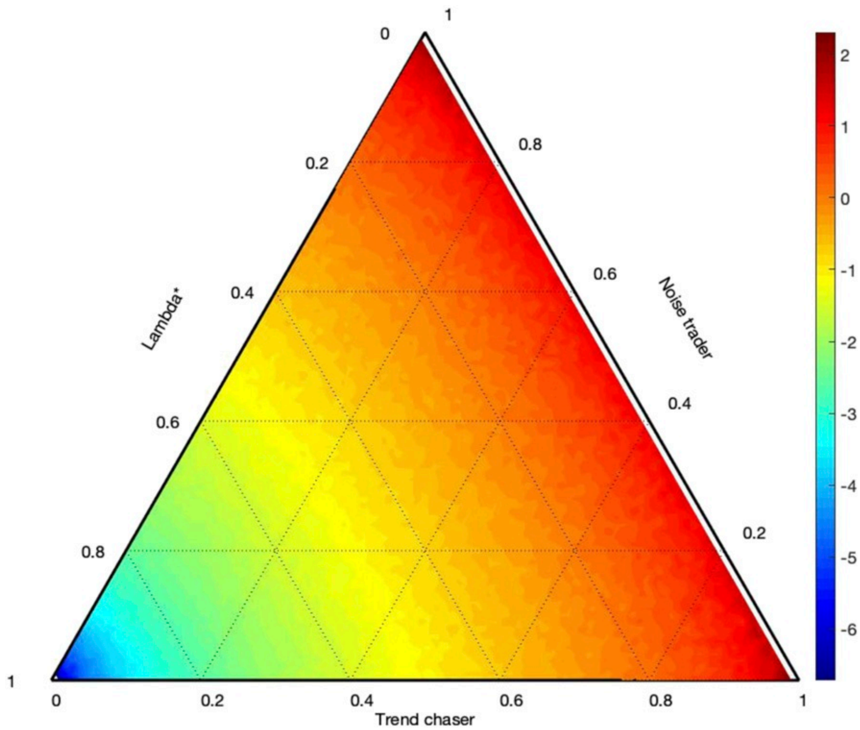

Fig. 3. Growth rate of λ^* investment strategy. We measure the expected growth of wealth over 20 periods for each combination of initial wealth across the three strategies.

each period, prices for both assets are computed by Eq. **11**, as well as the ratio of prices $p_t := p_{t,1}/p_{t,2}$. Then we calculate $|p^* - p_t|$ where $p_k^* = \frac{\rho}{V_k}\lambda_k^*$ are the equilibrium prices (the prices that prevail when the ESS λ^* has all wealth, i.e., $r^1 = 1$). Portfolios are defined by Eq. **8**. Additionally, we calculate fractions of total market wealth allocated to assets by Eq. **3**. Next, we generate random state of the world s_t and calculate dividends $d_k(s_t, W_{t-1,k})$. Afterward, we calculate the new wealth of all strategies by Eq. **6** as well as returns and relative dividends, which are needed for the trend chaser.

Fig. 2 illustrates the number of model iterations to obtain prices close to those prevailing in the long-run equilibrium. We calculate the expected number of periods to obtain $|p^* - p_t| < 2.5\%$, where p is the relative price of the two assets (of course, full equilibrium is attained only asymptotically). Each point in the diagram corresponds to a distribution of wealth across the three strategies. First, we observe that the equilibrium is indeed stable. Even for large perturbations, prices revert to equilibrium. Second, it requires a large deviation from equilibrium to have long-lasting mispricing. For instance, when the noise trader acquires 10% of wealth, it takes 12 iterations of the wealth dynamic to return to equilibrium prices. For the trend chaser, 4% of total wealth has the same effect.

One can also determine which strategy has the highest growth rate of wealth for each particular initial distribution of wealth. It turns out that λ^* has the highest expected growth for all initial distributions. As an implication it follows that λ^* is globally stable against the history-dependent momentum strategy and the noise trader. Fig. 3 shows that the wealth of λ^* grows the fastest, the poorer the strategy. The composition of the market ecology (moving parallel to the vertex of the noise trader) has no impact on λ^*'s growth rate. One needs to take into account that the wealth dynamics is driven solely by capital gains. There is no exogenous flow of capital from worse to better performing strategies. The result in Fig. 3 indicates that if such a flow were present, convergence to equilibrium prices would be even faster, enhancing the stability of the ESS.

Discussion

Financial markets are modeled from a biological perspective where investment strategies and wealth take on the role of species and their fitness. The main innovation is that dividends paid by a firm's stocks are determined by the firm's market value and random events. This creates a feedback loop between the wealth distribution and the production of dividends that are paid to the investment strategies. We analyze the resulting stochastic dynamical system with the aim to determine whether there are investment strategies such that the wealth distribution is (locally) stable.

Our main result is the explicit description of a unique evolutionary stable investment strategy λ^* such that the state in which this strategy has all of the wealth is locally stable. Invading investment strategies will be driven out by the wealth dynamics. The remarkable property of λ^* is that it only depends on the dividend production function and not on the ecology of the financial market. It has several interesting properties such as full diversification, which is of importance in portfolio management, and the valuation of financial assets in terms of relative fundamentals, which matters in asset pricing.

Our numerical results suggest that λ^* might be globally stable. A drawback of the current model is the restriction to i.i.d. shocks, which allow us to work with fix-mix investment strategies. Future work will aim at relaxing this assumption to Markovian dividends and strategies or possibly to an arbitrary time- and history-dependent setting, which has been done successfully for exogenous dividend processes.

Data Availability All study data are included in the article and *SI Appendix*.

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