POWER SERIES EXPANSIONS IN THE NEIGHBORHOOD OF A POINT ON A SURFACE

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1. *Introduction.*—Starting with a system of differential equations which define a surface except for a projective transformation, we present a general method of calculating the coefficients of a canonical expansion of one non-homogeneous projective coördinate of a point on the surface as a power series in the other two coördinates to any number of terms. And we compute for the first time the coefficients of the terms of the fifth order. Then several canonical expansions obtained to terms of the fourth order by Darboux, Wilczynski, Green, Fubini and others are correlated and a general expansion is obtained to terms of the fifth order, which, by suitable specializations combined with certain minor modifications, yields all of these expansions. Finally some geometric applications of the analytic results are made.

2. *Computation.*—If the four homogeneous projective coördinates $x^{(1)}$, ..., $x^{(4)}$ of a point P on a non-degenerate non-ruled surface S in ordinary space are given as analytic functions of two independent variables u, v, and if the parametric net on S is the asymptotic net, then the functions x are solutions of a system of differential equations which, by suitable choice of proportionality factor, can be reduced to Fubini's canonical form

$$x_{uu} = px + \theta_u x_u + \beta x_v, \quad x_{vv} = qx + \gamma x_u + \theta_v x_v, \quad \theta = \log(\beta\gamma). \quad (1)$$

The coördinates of a point Q on S near P may be represented by Taylor's formula as power series in the increments Δu, Δv corresponding to displacement from P to Q. Then by means of equations (1) and the equations obtained therefrom by differentiation it is possible to express each of these series in the form $x_1 x + x_2 x_u + x_3 x_v + x_4 x_{uv}$, where $x_1, \ldots, x_4$ are power series in Δu, Δv which represent the local coördinates of Q referred to the covariant tetrahedron x, x_u, x_v, x_{uv}, with a suitably chosen unit point.

It is easy to compute power series in Δu, Δv for the non-homogeneous coördinates x, y, z of Q defined by placing $x = x_2/x_1$, $y = x_3/x_1$, $z = x_4/x_1$. From these series it is possible to compute an expansion for z as a power series in x, y. The best method of procedure seems to be to set z equal to a power series in x, y with undetermined coefficients and then to demand that the expansions for x, y, z in powers of Δu, Δv shall satisfy this equation identically in Δu, Δv as far as terms of any desired order, starting with the terms of order zero, and then determining the coefficients of the terms of orders one, two, three, etc., sequentially.

The result of carrying out the computations indicated, to terms of order five, is

$$z = xy - \frac{1}{3}(\beta x^3 + \gamma y^3) + \frac{1}{12}[\beta\psi_1 x^4 - 4\beta\psi_2 x^3 y - 6\theta_{uv} x^2 y^2 - 4\gamma\psi_1 x y^3 + \gamma\psi_2 y^4] + \frac{1}{60}[\beta c_0 x^5 + 5\beta c_1 x^4 y + 10\beta c_2 x^3 y^2 + 10\gamma c_3 x^2 y^3 + 5\gamma c_4 x y^4 + \gamma c_5 y^5] + \ldots \quad (2)$$

where

$$\psi_1 = \frac{\partial}{\partial u} \log \beta\gamma^2, \qquad \psi_2 = \frac{\partial}{\partial v} \log \beta^2\gamma$$

$$c_0 = \psi_1 \frac{\partial}{\partial u} \log \frac{\psi_1}{\beta\gamma} - \psi_1^2 + 8\beta\psi_2 - 4p, \quad c_1 = \psi_1\psi_2 - \psi_{2u} + 7\theta_{uv} + 4\beta\gamma,$$

$$c_2 = 3\gamma\psi_1 - \psi_2 \frac{\partial}{\partial v} \log \beta\psi_2 + 2q, \qquad c_3 = 3\beta\psi_2 - \psi_1 \frac{\partial}{\partial u} \log \gamma\psi_1 + 2p, \quad (3)$$

$$c_4 = \psi_1\psi_2 - \psi_{1v} + 7\theta_{uv} + 4\beta\gamma, \qquad c_4 = \psi_2 \frac{\partial}{\partial v} \log \frac{\psi_2}{\beta\gamma} - \psi_2^2 + 8\gamma\psi_1 - 4q.$$

If a covariant unit point is introduced by the transformation

$$x = l\bar{x}, \quad y = m\bar{y}, \quad z = n\bar{z},$$

where

$$l = \epsilon/\sqrt[3]{\beta^2\gamma}, \quad m = \epsilon^2/\sqrt[3]{\beta\gamma^2}, \quad n = 1/\beta\gamma, \quad \epsilon^3 = 1,$$

the expansion (2) becomes

$$z = xy - \frac{1}{3}(x^3 + y^3) + \frac{1}{12}[a_0 x^4 + 4a_1 x^3 y + 6a_2 x^2 y^2 + 4a_3 x y^3 + a_4 y^4] + \frac{1}{60}[b_0 x^5 + 5b_1 x^4 y + 10b_2 x^3 y^2 + 10b_3 x^2 y^3 + 5b_4 x y^4 + b_5 y^5] + \ldots, \quad (4)$$

where

$$a_0 = l\psi_1, \quad a_1 = -m\psi_2, \quad a_2 = -n\theta_{uv}, \quad a_3 = -l\psi_1, \quad a_4 = m\psi_2, \quad b_0 = mc_0/\beta, \quad b_1 = nc_1, \quad b_2 = lc_2/\gamma, \quad b_3 = mc_3/\beta, \quad b_4 = nc_4, \quad b_5 = lc_5/\gamma. \quad (5)$$

The new unit point, which in the original local system has coördinates $(1, l, m, n)$, may be characterized geometrically. This point lies on Fubini's canonical quadric,[1] $x_2x_3 - x_1x_4 = 0$, and lies on one of the three planes

$$\epsilon\beta^{1/3}x_2 - \gamma^{1/3}x_3 = 0,$$

determined by the projective normal and a tangent of Segre. And if the point be projected from the point (0, 0, 0, 1) onto the tangent plane

$x_4 = 0$, the projection, which will be denoted by U, has coördinates $(1, l, m, 0)$. The characterization will be completed by showing how to locate U by means of the cross ratio formed by U with three other points on the tangent of Segre. One of these is the point P with coördinates $(1, 0, 0, 0)$. A second is the point R, with coördinates

$$(0, \quad \gamma^{1/3}, \quad \epsilon\beta^{1/3}, \quad 0),$$

where the Segre tangent crosses the polar $x_1 = x_4 = 0$ of the projective normal $x_2 = x_3 = 0$ with respect to the canonical quadric. The third is the point T, with coördinates

$$[2\epsilon^2(\beta\gamma)^{2/3}, \quad \gamma^{1/3}, \quad \epsilon\beta^{1/3}, \quad 0],$$

where the Segre tangent crosses the cubic curve

$$x_4 = \beta x_2^3 + \gamma x_3^3 - x_1x_2x_3 = 0$$

corresponding to the projective normal by Segre's transformation. The point U is such that $(P, R, U, T) = 2$.

3. *Correlation of Different Expansions.*—The fact that two opposite edges of the tetrahedron of reference for the expansion (4) are the projective normal and its polar with respect to the canonical quadric of Fubini suggests that we may use instead any other two lines in the same polar relation, provided that one of them passes through P but does not lie in the tangent plane π, while the other lies in π but does not pass through P. The transformation which effects this change of tetrahedron is

$$x = (\bar{x} - A\bar{z})/D, \quad y = (\bar{y} - B\bar{z})/D, \quad z = \bar{z}/D, \quad D = 1 - B\bar{x} - A\bar{y} + h\bar{z},$$

the new unit point being related to the new edges of the new tetrahedron in the same way as the old unit point was related to the projective normal and its polar. Under this transformation the expansion (4) becomes

$$z = xy - \frac{1}{3}(x^3 + y^3) + \frac{1}{12}[(a_0 - 4B)x^4 + 4(a_1 + 2A)x^3y + 6(a_2 + 2AB - 2h)x^2y^2 + 4(a_3 + 2B)xy^3 + (a_4 - 4A)y^4] + \frac{1}{60}[(b_0 + 10a_0B - 20B^2 - 20A)x^5 + 5(b_1 + 12h - 2a_0A - 4AB + 4a_1B)x^4y + 10(b_2 - 2a_1A - 2A^2 - 2B)x^3y^2 + 10(b_3 - 2a_3B - 2B^2 - 2A)x^2y^3 + 5(b_4 + 12h - 2a_4B - 4AB + 4a_3A)xy^4 + (b_5 + 10a_4A - 20A^2 - 20B)y^5] + \ldots \quad (6)$$

If the two reciprocal polar lines which we have just chosen for edges of the fundamental tetrahedron are canonical lines,[2] then

$$A = -mk\psi_2, \quad B = -lk\psi_1, \quad k = \text{const.},$$

and the expansion (6) becomes

$$z = xy - \frac{1}{3}(x^3 + y^3) + \frac{1}{12}[(1 + 4k)l\psi_1 x^4 - 4(1 + 2k)m\psi_2 x^3 y + 6(nk^2\psi_1\psi_2 - n\theta_{uv} - 2h)x^2y^2 - 4(1 + 2k)l\psi_1 xy^3 + (1 + 4k)m\psi_2 y^4] + \frac{1}{60}\Big[\Big\{\psi_1 \frac{\partial}{\partial u}\log\frac{\psi_1}{\beta\gamma} - (20k^2 + 10k + 1)\psi_1^2 + 4(2 + 5k)\beta\psi_2 - 4p\Big\}l^2x^5 + 5\{(1 + 2k)(1 + 4k)\psi_1\psi_2 - \psi_{2u} + \theta_{uv} + 4\beta\gamma\}nx^4y + 10\Big\{(3 + 2k)\gamma\psi_1 - \psi_2\frac{\partial}{\partial v}\log\beta\psi_2 + 2q - 2k(1 + k)\psi_2^2\Big\}m^2x^3y^2 + 10\Big\{(3 + 2k)\beta\psi_2 - \psi_1\frac{\partial}{\partial u}\log\gamma\psi_1 + 2p - 2k(1 + k)\psi_1^2\Big\}l^2x^2y^3 + 5\{(1 + 2k)(1 + 4k)\psi_1\psi_2 - \psi_{1v} + \theta_{uv} + 4\beta\gamma\}nxy^4 + \Big\{\psi_2\frac{\partial}{\partial v}\log\frac{\psi_1}{\beta\gamma} - (20k^2 + 10k + 1)\psi_2^2 + 4(2 + 5k)\gamma\psi_1 - 4q\Big\}m^2y^5\Big] + \ldots \quad (7)$$

If the vertex (0, 0, 0, 1) be chosen so as to lie on the canonical quadric of Wilczynski, whose equation is

$$x_2x_3 - x_1x_4 + \left(AB - \frac{1}{2}\frac{\theta_{uv}}{\beta\gamma} - h\right)x_4^2 = 0,$$

then the coefficient of x^2y^2 in expansion (6) vanishes. With this simplification and with the transformation of unit point

$$x = -\frac{1}{2}\bar{x}, \quad y = -\frac{1}{2}\bar{y}, \quad z = \frac{1}{4}\bar{z},$$

equation (7) is Wilczynski's canonical expansion[3] if $1 + 2k = 0$, and is Green's canonical expansion[4] if $1 + 4k = 0$. These geometers, however, carried their expansions only to terms of the fourth order and started with a different canonical form of the fundamental differential equations.

If we had started with the canonical form of the differential equations characterized by the conditions $p = q = 0$, then equation (7) would contain all of Fubini's canonical expansions.[5] He carries his expansions only to terms of the fourth order, however.

4. *Geometric Applications.*—We shall now recur to expansion (2) and make some geometric applications thereof, remarking that similar applications might be made of any of our other expansions.

First of all, we shall obtain conditions on the coefficients of equations (1) which are necessary and sufficient that S be a cubic. If we write the most general equation in x, y, z of a non-composite cubic surface and then seek to determine the coefficients of this equation so that the cubic may have contact of the fifth order with S at P, we find that S must be itself a cubic, and that the required conditions, therefore, are

$$\begin{aligned} \frac{\partial^2}{\partial u \partial v} \log \frac{\beta}{\gamma} &= 0, \\ p &= \frac{1}{4}\psi_1 \frac{\partial}{\partial u} \log \frac{\psi_1}{\beta\gamma} + \frac{1}{16}\psi_1^2 - \frac{1}{12}\beta\psi_2, \\ q &= \frac{1}{4}\psi_2 \frac{\partial}{\partial v} \log \frac{\psi_2}{\beta\gamma} + \frac{1}{16}\psi_2^2 - \frac{1}{12}\gamma\psi_1. \end{aligned} \tag{8}$$

The algebraic surface of order n obtained by truncating the series (2) after terms of degree n will be called the canonical n-ic surface, and will be denoted by S_n. This is Fubini's canonical quadric if $n = 2$, and is his canonical cubic if $n = 3$. The general surface S_n may be characterized geometrically by two properties. First, S_n has a unode of order $n - 1$ at the point $(0, 0, 0, 1)$, the unodal plane being the plane $x_1 = 0$; this means that the tangent plane and the polar surfaces of S_n at $(0, 0, 0, 1)$ up to and including the surface of order $n - 2$ are indeterminate, while the polar surface of order $n - 1$ degenerates into the plane $x_1 = 0$ counted $n - 1$ times. Second, S_n has contact of order n with S at P.

If we denote by φ_k the sum of terms of degree k in (2), then the straight lines $x_4 = \varphi_n = 0$ are the tangents at P to the curve of intersection of S_{n-1} and S. The planes $\varphi_n = 0$ project these tangents from $(0, 0, 0, 1)$ and, moreover, contain the entire intersection of S_{n-1} and S_n which, therefore, consists of n plane curves of order $n - 1$.

More generally, S_{k-1} and S_n intersect on a cone which has its vertex at $(0, 0, 0, 1)$ and which intersects the tangent plane in a curve with a k-ple point at P, the k-ple point tangents being the lines $x_4 = \varphi_k = 0$. We shall denote this curve by $C_{k-1, n}$. In particular, $C_{1,n}$ is the intersection of S_n and the tangent plane, and $C_{1,2}$ consists of the asymptotic tangents. The curve $C_{n-2, n}$ seems to merit special consideration and can be characterized independently as follows. It is a curve of order n in the tangent plane with an $(n - 1)$-ple point at P, the $(n - 1)$-ple point tangents being the lines $x_4 = \varphi_{n-1} = 0$. It, moreover, passes through the intersections of the reciprocal of the projective normal, $x_4 = x_1 = 0$, with the planes $\varphi_n = 0$. The most general curve with these two properties has an equation of the form

$$x_4 = x_1\varphi_{n-1} + h\,\varphi_n = 0, \qquad h = \text{const.} \tag{9}$$

For $C_{n-2, n}$, h has such a value that the curve passes through the intersections of $C_{1, n-2}$ and $C_{1, n}$.

It is easy to show that each of the curves (9) crosses the asymptotic tangents in two points, and that the line joining these points intersects $x_4 = x_1 = 0$ in a point that is independent of h, so that all of these lines form a pencil. In particular, if $n = 5$, the coördinates of the center of this pencil are $(0, -\psi_1 c_5, \psi_2 c_0, 0)$, and if $n = 4$ the center of the pencil is the canonical point $(0, -\psi_2, \psi_1, 0)$.

We have occasionally tacitly supposed $\psi_1\psi_2 \neq 0$. If $\psi_1 = \psi_2 = 0$, the surface S is such that all canonical lines through P coincide with the projective normal. Moreover, by a change of parameters we can make $\beta = \gamma = 1$. Then it is easy to see that S_3 has fourth order contact with S at P. Conversely, if S_3 has fourth order contact, and if S is not ruled, then $\psi_1 = \psi_2 = 0$.

[1] Fubini e Čech, *Geometria Proiettiva Differenziale*, Vol. I, p. 129.

[2] Fubini e Čech, loc. cit., p. 155.

[3] Wilczynski, "Second Memoir," *Trans. Amer. Math. Soc.*, **9** (1908), p. 98.

[4] Green, "Memoir on the General Theory of Surfaces, Etc.," *ibid.*, **20** (1919), p. 108.

[5] Fubini, Nuova trattazione elementare dei fondamenti della geometria proiettivo—differenziale di una superficia, *Rend. Reale Istituto Lombardo*, **59** (1926), p. 69.

THE ANALYSIS AND ANALYSIS SITUS OF REGULAR n-SPREADS IN (n + r) – SPACE

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The present note consists of a set of theorems and statements of methods, the elaboration of which will appear later in some other place.

1. *The Existence of Non-Degenerate Functions on n-Spreads without Boundary.*—We shall deal with n-spreads defined as follows. Let $(y) = (y_1, \ldots, y_{n+r})$ be rectangular coördinates in an euclidean $(n + r)$-space. $n > 1$, $r > 0$. Let $(u) = (u_1, \ldots, u_n)$ be rectangular coördinates in an auxillary n-space. By an *element* of an n-spread in our $(n + r)$-space will be meant a set of points homeomorphic with a set of points (u) within an $(n - 1)$ sphere in the space (u). By an *n-spread* $\sum_n$ *without boundary* will be meant a closed set of points (y), each one of which is on at least one of a finite set of elements and all points of which neighboring a given point of $\sum_n$ can be included in a single element of $\sum_n$.

An element will be said to be of *Class* C^n if the y's as functions of the u's are of class C^n, that is, possess continuous nth order partial derivatives. An element will be called *regular* if it is of at least class C', and at least one of the jacobians of n of the y's with respect to the u's is not zero, while $\sum_n$ will be called regular if capable of a representation by regular