2.56% in X-rayed seeds and 0.08% in untreated seeds. If these average frequencies, determined from five paternal chromosomes (but chiefly from chromosomes II and III), may be taken to represent roughly the frequency of loss of the 30 chromosomes present in the triploid endosperm nucleus, an average rate of about 75 aberrancies per 100 mitoses is indicated for the first division in endosperm development in the irradiated seeds.

[1] Formerly NATIONAL RESEARCH FELLOW. Part of the work reported was done under this fellowship at the Bussey Institution, Harvard University, 1925–26.

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GENERAL THEORY OF POLYGENIC OR NON-MONOGENIC FUNCTIONS. THE DERIVATIVE CONGRUENCE OF CIRCLES

BY EDWARD KASNER

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY

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This paper is devoted mainly to the geometric questions connected with the differentiation of a general complex function $w = \varphi(x, y) + i\psi(x, y)$ with respect to the independent variable $z = x + iy$. We assume merely that the components φ and ψ are continuous and have continuous partial derivatives in the region considered, but do not assume the Cauchy-Riemann equations.

The limit of the increment ratio $\Delta w/\Delta z$, in general, then depends not only on the point $x + iy$ but also on the direction θ (or the slope m) along which the neighboring point approaches the given point. Thus the derivative dw/dz has, in general, infinitely many complex values at a given point. We call such a function *polygenic*. Only when the Cauchy-Riemann equations are fulfilled will the derivative have a unique value at the point; the function is then called *monogenic*. Thus the derivative of a polygenic function is a function of z and m while the derivative of a monogenic function is a function of z alone.

I show in the following that the ∞^1 values of dw/dz at each point z can be plotted as a circle and study the congruence of circles corresponding to the ∞^2 values of z. This congruence reduces to points only for monogenic functions.

I also introduce the new notion of *mean derivative*. It is used to define a certain class of harmonic transformations determined by polygenic functions whose components satisfy the Laplace equation without fulfilling the Cauchy-Riemann conditions.

Notation.—The notation will be as follows:

For the independent variable we use

$$z = x + iy,$$

for the dependent variable

$$w = u + iv,$$

and for the corresponding Gaussian planes (z) and (w). To indicate that w is a general polygenic function of z we write

$$w = \mathfrak{F}(z),$$

and reserve the familiar symbol

$$w = f(z)$$

for monogenic functions. Likewise

$$w = f(\bar{z})$$

shall denote an analytic function of $(x-iy)$ in the customary sense. Since the totality of polygenic functions contains the monogenic ones and the function of $\bar{z}$ as special cases, $\mathfrak{F}(z)$ can, under certain circumstances, specialize into $f(z)$ or $f(\bar{z})$.

For the real and imaginary components of $\mathfrak{F}(z)$ we take φ and ψ, so that

$$\mathfrak{F}(z) = \varphi(x, y) + i\psi(x, y).$$

For the sake of easier reference we insert here the Cauchy-Riemann equations and the corresponding conditions for functions of $(x-iy)$:

$$\begin{array}{lll} w = f(z): & \qquad w = f(\bar{z}): & \\ \varphi_x - \psi_y = 0 & \qquad \varphi_x + \psi_y = 0 & \qquad (1) \\ \varphi_y + \psi_x = 0 & \qquad \varphi_y - \psi_x = 0. & \end{array}$$

The transformation of (z) into (w) established by $w = \mathfrak{F}(z)$ will be called T; the transformation formulas are

$$u = \varphi(x, y).$$
$$v = \psi(x, y)$$

If we differentiate w with respect to z we have

$$\frac{dw}{dz} = \frac{(\varphi_x + i\psi_x)\,dx + (\varphi_y + i\psi_y)dy}{dx + idy}.$$

We represent this expression, which is, of course, complex, by the symbol

$$\gamma = \alpha + i\beta.$$

Putting m for the slope dy/dx, we have finally

$$\gamma = \frac{\varphi_x + i\psi_x + m(\varphi_y + i\psi_y)}{1 + im}.$$

In the following we shall deal with the Gaussian planes of the three complex quantities

$$z = x + iy, \qquad w = u + iv, \qquad \gamma = \alpha + i\beta;$$

they will be abbreviated by (z), (w), (γ), and referred to as the first, second and third plane in this order. The third plane will frequently also be referred to as the derivative plane.

The Derivative Circle.—A polygenic function w of z sets up a correspondence between the first and third plane which is expressed by

$$\alpha + i\beta = \frac{\varphi_x + i\psi_x + m(\varphi_y + i\psi_y)}{1 + im}.$$

To a general point $x + iy$ belong ∞^1 points $\alpha + i\beta$, corresponding to the real parameter m. Since we are dealing with a linear function of m, these points are in one-to-one correspondence with the slopes m. For the same reason, the points must obviously form a circle. We call it the *derivative circle*. As we need its explicit equation in the following, we actually carry out the elimination of m and α and β. We have

$$\alpha = \frac{\varphi_x + m(\varphi_y + \psi_x) + m^2\psi_y}{1 + m^2}, \quad \beta = \frac{\psi_x + m(\psi_y - \varphi_x) - m^2\varphi_y}{1 + m^2} \tag{2}$$

$$\alpha - \psi_y = -\frac{1}{m}(\beta - \psi_x), \quad \alpha - \varphi_x = m(\beta + \varphi_y), \tag{3}$$

and so for the final equation

$$(\alpha - \psi_y)(\alpha - \varphi_x) + (\beta - \psi_x)(\beta + \varphi_y) = 0,$$

which may be rewritten

$$\left(\alpha - \frac{\varphi_x + \psi_y}{2}\right)^2 + \left(\beta - \frac{-\varphi_y + \psi_x}{2}\right)^2 = \left(\frac{\varphi_x - \psi_y}{2}\right)^2 + \left(\frac{\varphi_y + \psi_x}{2}\right)^2. \tag{4}$$

The ∞^1 *points* $\alpha + i\beta$ *mapping the* ∞^1 *values assumed by the derivative* dw/dz *at a fixed point* z *form a circle; we call it the "derivative circle." Denoting by* H *and* K *its center coördinates and by* R *its radius, we have*

$$H = \frac{\varphi_x + \psi_y}{2}, K = \frac{-\varphi_y + \psi_x}{2}$$

$$R = \tfrac{1}{2}\sqrt{(\varphi_x - \psi_y)^2 + (\varphi_y + \psi_x)^2}.$$

We notice that these three quantities are essentially made up of the expressions on the left-hand sides of the formulas (1).

We also remark that the derivative circle may degenerate into a point but never into a straight line.

The Congruence of Circles.—A polygenic function establishes a relation between the first and third plane in which to any point of the former corresponds a circle in the latter; to the ∞^2 distinct points of (z) will, therefore, in general correspond ∞^2 distinct circles in (γ). A configuration of ∞^2 curves is termed a congruence.

Thus the derivative of any polygenic function w *is (in general) represented by a congruence of circles.*

The limiting term "in general" in this theorem refers to "circles" as well as to "congruence;" it implies that degenerations in either respect are possible. Two very familiar cases furnish examples.

I. For monogenic function the derivative circles all reduce to points, since R vanishes identically (on account of the Cauchy-Riemann equations); and conversely if all the circles reduce to points the function must be monogenic.

II. For analytic functions of $(x-iy)$, H and K are identically zero and the double infinity of circles, therefore, reduces to a single infinity with the fixed center (0, 0); conversely if the ∞^2 circles are to reduce to ∞^1 with the common center (0, 0), w must be of the form $f(x-iy)$.

The whole congruence can also reduce to a single circle. We merely state this here, since we shall be able to give the discussion more simply in a later paper.

In general, any special property of a polygenic function is expressed by a corresponding property of the representative congruence; and vice versa, special types of polygenic functions are obtained by specifying special geometric types of the congruence. We give some further examples.

III. Determine all functions w such that at every point of (z) the modulus of the derivative is constant with regard to m. The derivative circles must then obviously reduce to points or have the fixed center (0, 0). So by the converse theorems of I and II the required functions are of the forms $w = f(z)$ and $w = f(\bar{z})$.

IV. Similarly, the only functions for which the amplitude of the derivative at every point of (z) is independent of m are recognized, by the converse of I, to be the monogenic functions.

V. Require all the derivative circles to go through the origin. The analytic condition is found to be

$$\varphi_x\psi_y - \varphi_y\psi_x = 0,$$

that is, the required functions are characterized by the vanishing of the Jacobian of φ and ψ.

VI. Let the centers of all the circles lie on the α-axis. Then, as K vanishes, $-\varphi_y + \psi_x = 0$, so that

$$\varphi = W_x \quad , \quad \psi = W_y,$$

that is,

$$w = W_x + iW_y,$$

where W is an arbitrary function of x and y. Similarly, when all the centers lie on the β-axis, we find

$$w = -W_y + iW_x.$$

VII. Require all the centers to lie in a general fixed point. The corresponding functions are readily found to be

$$w = f(x - iy) + Ax + By + C,$$

where A, B, C are any complex constants.

We shall here introduce the term *affine* for the linear part of w, since later we will often have to deal with expressions of the same type. An "affine" then is an expression of the form

$$Ax + By + C,$$

where A, B, C are complex constants.

VIII. Represent z and γ in the same plane and require the points z and the centers of the respective derivative circles to coincide. The analytic condition is

$$\frac{\varphi_x + \psi_y}{2} = x, \quad \frac{-\varphi_y + \psi_x}{2} = y.$$

The complete solution is given by

$$w = \frac{z^2}{2} + f(x - iy).$$

The corresponding problem arising when w and γ are mapped in the same plane can also be solved in finite form; the result is

$$w = e^z f(x - iy).$$

The Ratio $(-2{:}1)$ *of the Angular Rates.*—After having become somewhat familiar with the congruence as a whole by the previous examples, let us now return again to a definite derivative circle corresponding to a definite point z and inquire how $\alpha + i\beta$ moves on the circle when m changes. The answer is given by the first formula of (3):

$$m = -\frac{\beta - \psi_x}{\alpha - \psi_y}$$

after verifying that the point $\psi_y + i\psi_x$ lies on the circle. For $\frac{\beta - \psi_x}{\alpha - \psi_y}$ is the slope of the line through $\psi_y + i\psi_x$ and $\alpha + i\beta$; according to the above equation, it is equal in magnitude and opposite in sign to the corresponding slope m. So we have a very simple method for constructing the point belonging to a certain m: it is the second intersection with the circle of the line of slope $-m$ through $\psi_y + i\psi_x$. If in this manner we construct two points P_1 and P_2 corresponding to the slopes m_1 and m_2, we see that the angle at the circumference over the arc P_1P_2 (since $\psi_y + i\psi^x$ lies on the circle) is equal in size to the angle between m_1 and m_2. Therefore, the cental angle over the same arc is twice as large as the original angle in (z), and we have the theorem:

When m changes, the point on the derivative circle moves so that its angular rate (measured on the circle) is twice that of m and in the opposite direction.

(It is obvious that on the circle the same point corresponds to two directions in (z) that differ by π; it is not obvious, however, that the ratio of the angular rates of m and the point on the circle should be constant.)

Thus each circle must be replaced by what I have called a "clock," and we must study *congruences of clocks*. See my paper in *Science*, vol. **66** (1927), pp. 281–282.

The Mean Derivative Operation $\mathfrak{D}$.—In order to be able to formulate some of the later results in a simple manner, we introduce a new term: we call the complex quantity $H + iK$ corresponding to the center of the derivative circle the *mean derivative* of w with regard to z and use for it the symbol

$$\mathfrak{D}(w).$$

Thus

$$\mathfrak{D}(w) = H + iK,$$

$$\mathfrak{D}(\varphi + i\psi) = \frac{\varphi_x + \psi_y}{2} + i\,\frac{-\varphi_y + \psi_x}{2}. \tag{5}$$

The process of forming the mean derivative of $w = \psi + i\varphi$ consists in forming the expression on the right-hand side of (5). Of course, $\mathfrak{D}(w)$ is also a polygenic function of z.

We also introduce the symbol $\mathfrak{D}^{-1}$; it stands for the operation inverse to that of forming the mean derivative. This inverse process gives a result which is determined not up to an additive constant but up to an additive term $f(x-iy)$.

The term "mean derivative" as introduced above actually agrees with the usual conception of "mean value." The mean value of $\alpha + i\beta$ considered as a function of m in a definite point z is in the customary sense if we put $m = \tan\theta$,

$$\frac{1}{\pi}\int_0^{\pi}(\alpha + i\beta)d\theta.$$

Carrying out the integration, *we actually obtain the expression on the right-hand side of* (5).

A polygenic function induces a certain point-transformation from the first plane to the third plane, namely that transformation in which to a point $x + iy$ of (z) corresponds the point $\mathfrak{D}(w)$ of (γ). Since this transformation leads from z to the center of the corresponding derivative circle, we call it the *induced center transformation;* we denote it by T'. Hence for any point-transformation T there is a related T'.

Harmonic Polygenic Functions and Harmonic Transformations.—The familiar theorem of the ordinary theory of functions that the derivative of a monogenic function is also monogenic, reads in the new terminology: when w is monogenic the induced center transformation is direct conformal. In symbols

$$\mathfrak{D}f_1(z) = f_2(z).$$

Let us put the converse question and ask whether

$$\mathfrak{D}^{-1}f_1(z) = f_3(z),$$

i.e., whether the monogenic functions are the only functions whose mean derivative is monogenic? If $H + iK$ is to be monogenic, then

$$H_x - K_y = 0 \qquad H_y + K_x = 0;$$

from which we find

$$\varphi_{xx} + \varphi_{yy} = 0 \qquad \psi_{xx} + \psi_{yy} = 0.$$

The necessary and sufficient condition to be fulfilled by $w = \varphi + i\psi$ in order that the induced center transformation may be direct conformal or, differently expressed, that the mean derivative of w may be monogenic, is that φ and ψ are harmonic functions.

We thus obtain a geometric interpretation for the case where the Laplace equations are obeyed but not necessarily the Cauchy-Riemann equations.

According to familiar theorems, w then has the form

$$w = f_1(z) + f_2(\bar{z}).$$

We term this a *harmonic polygenic function.* The corresponding point transformation from the z-plane to the w-plane will be called a *general harmonic transformation.* The set of all such transformations does not form a group but it contains the group of all conformal transformations (direct and reverse) as a part.

Forthcoming papers on polygenic functions will appear in *Comptes Rendus* and *Bull. Amer. Math. Soc.*

NUMBER OF SYSTEMS OF IMPRIMITIVITY OF TRANSITIVE SUBSTITUTION GROUPS

By G. A. Miller

Department of Mathematics, University of Illinois

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It is well known that the number of the systems of imprimitivity of a regular substitution group is equal to the number of its proper subgroups, that is, the number of its subgroups excluding the identity and the entire group. This is a special case of the following general theorem: *The number of the systems of imprimitivity of any transitive substitution group is equal to the number of the proper subgroups of this group which separately involve as a proper subgroup a fixed one of the subgroups composed of all the substitutions of this transitive group which omit a given letter, and the number of letters in a set of such a system of imprimitivity is equal to the index of this fixed subgroup under the larger subgroup.* To prove this theorem it is only necessary to observe that if G_1 is composed of all the substitutions of a transitive group G which omit a given letter, and if G_1' is any proper subgroup of G which involves G_1 as a proper subgroup, then the letter which is omitted by all the substitutions of G_1 is transformed into a certain number of conjugates under G_1'. Moreover, G_1' involves all the substitutions of G which transform these letters among themselves since it involves all of these substitutions which transform the given omitted letter into itself. Hence these letters constitute a set of a system of imprimitivity of G.

On the other hand, when G is imprimitive, such a subgroup G_1' must exist for every possible system of imprimitivity since all the letters of G must appear in every one of its possible systems of imprimitivity. Each of the subgroups G_1' must, therefore, characterize the system of imprimitivity to