CONCERNING TRIODS IN THE PLANE AND THE JUNCTION POINTS OF PLANE CONTINUA

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Menger[1] and Alexandroff[2] use the term *verzweigungspunkt*, and Kuratowski and Zarankiewicz[3] the term *point of ramification*, to denote a point of Menger order greater than two of a regular curve. It has been shown, by Menger,[4] that if P is such a point of a regular curve C then C contains three simple continuous arcs such that P is common to all three of them and such that no two of them have any other point in common.

Wazewski[5] and Menger[1] have shown that the set of all the points of ramification of an acyclic[6] continuous curve is countable and Alexandroff[2] and Menger[1] have established the same result for a finitely connected continuous curve, that is to say for a continuous curve that does not contain more than a finite number of simple closed curves. It is clear, however, that this proposition does not[3] hold true for all regular curves.

In the present paper I shall introduce the notion *junction point of a continuum* and shall establish the theorem that no plane continuum whatsoever has more than a countable number of junction points. In the presence of the easily verified fact that every point of ramification of a regular curve of finite connectivity is also a junction point of that curve, the above-mentioned results of the authors cited are (insofar as they apply to the plane) special cases of this much more general theorem.

DEFINITION.—If O, A_1, A_2 and A_3 are four distinct points, and for each n $(1 \leqq n \leqq 3)$, r_n is an irreducible continuum from A_n to O and no two of the continua r_1, r_2 and r_3 have, in common, any point except O then the continuum $r_1 + r_2 + r_3$ is a *triod*, the point O is an *emanation point*, and the continua r_1, r_2 and r_3 are *rays* of this triod.

It may be easily shown that no triod has more than one emanation point or more than one set of three rays. It is, therefore, permissible to speak of *the* emanation point and *the* rays of a given triod. If O is a point and M_1, M_2 and M_3 are three continua which have O in common and no two of them have in common any point except O then the continuum $M_1 + M_2 + M_3$ contains at least one triod whose emanation point is O.

LEMMA 1.—*If, in space of two dimensions, J is a simple closed curve and t_1, t_2 and t_3 are three mutually exclusive arcs lying on J, and H and K are triods which are subsets of the point set composed of J plus its interior and h_1, h_2 and h_3 are the rays of H, and k_1, k_2 and k_3 are the rays of K and, for each n $(1 \leqq n \leqq 3)$, t_n contains a point of h_n and a point of k_n, then the triods H and K have at least one point in common.*

Lemma 1 may be proved with the help of various theorems established by Janiszewski,[7] Rosenthal[8] and Miss Mullikin.[9]

THEOREM 1.—*If, in space of two dimensions, G is an uncountable set of triods, there exists an uncountable subset of G such that every two triods of this subset have a point in common.*

Proof.—There exists a positive number d and an uncountable subset G_1 of G such that all three rays of every triod of the set G_1 are of diameter greater than d. Let O denote a point which is an emanation point of some triod of the set G_1 and a point of condensation of the set of all the points of emanation of the triods of G_1. Let K denote a circle with center at O and a radius equal to $d/2$. Let G_2 denote the set of all those triods of G_1 whose emanation points lie within K. For each triod t of the set G_2 select a definite triod q_t with the same emanation point as t and such that (a) q_t is a subset of t, (b) each ray of q_t is[9] an irreducible continuum from the emanation point of t to K. Let Q denote the set $[q_t]$ for all triods t of G_2. For each triod q of the set Q select three definite points A_q, B_q and C_q, one on each ray of q, and all lying on K. There exists a positive number e and an uncountable subset Q_1 of Q such that if q is any triod of the set Q_1 then every two of the points A_q, B_q and C_q are at a distance apart greater than e. It follows that there exist, on K, three distinct points A, B and C such that if x is any positive number there are uncountable many triods q belonging to Q_1 and such that the points A_q, B_q and C_q are at distances less than x from the points A, B and C, respectively. Let a, b and c denote three mutually exclusive arcs of K with midpoints at A, B and C, respectively. The set Q_1 contains an uncountable subset Q_2 such that if q is any triod of the set Q_2 then A_q, B_q and C_q belong to a, b and c, respectively. Every triod of the set Q_2 is a subset of K plus its interior. It follows, by Lemma 1, that every two triods of the set Q_2 have a point in common.

THEOREM 2.—*If, in space of two dimensions, G is an uncountable set of mutually exclusive bounded continuous curves then all but a countable number of them are either simple continuous arcs or simple closed curves.*

Theorem 2 may be easily proved with the aid of theorem 1 and the fact that if a bounded continuous curve contains no triod then it is either a simple continuous arc or a simple closed curve.

THEOREM 3.—*There exists, in space of two dimensions, an uncountable set G of mutually exclusive bounded continua such that no continuum of the set G contains an arc.*

Proof.—Knaster[10] has shown that there exists an indecomposable continuum which contains no decomposable subcontinuum. Let M denote such a continuum. By a theorem of Janiszewski and Kuratowski's,[11] M has an uncountable number of different composants. For each composant of M select just one continuum which is a subset of that composant. The set of continua so obtained is uncountable, no two of them have a point in common, and no one of them contains an arc.

THEOREM 4.—*If, in space of two dimensions, G is an uncountable set of mutually exclusive continuous curves then all but a countable number of them are simple*[12] *continuous curves.*

DEFINITION.—A point P of a continuum M is said to be a *junction point* of M if it is the emanation point of some triod which is a subset of M and there exists a domain D containing P and such that P is a cut point of the greatest connected subset of $M \cdot D'$ that contains P.

LEMMA 2.—*If P is a cut point of a continuum M and D is a domain containing P and H is a connected subset of M containing the greatest connected subset of $M \cdot D$ that contains P, then P is a cut point of H.*

The truth of this lemma may be easily established. It does not remain true if "continuum M" is replaced by "connected point set M."

THEOREM 5.—*No continuum has uncountably many junction points.*

Proof.—Suppose, on the contrary, that there exists a continuum M with an uncountable number of junction points. Then, as may be seen with the help of lemma 2, there exists a positive number d and an uncountable set K of junction points of M such that if P is a point of K and I_P is the interior of a circle with center at P and a radius equal to d and M_P is the greatest connected subset of $M \cdot I'_P$ that contains P then P is a cut point of M_P and M contains a triod having P as its emanation point. Let O denote a point of K which is a point of condensation of K and let D_O denote the interior of a circle with center at O and radius equal to $d/4$. For every point X of K lying in D_O let D_X denote the interior of a circle with center at X and radius equal to $d/4$ and let N_X denote the greatest connected subset of $M \cdot D'_X$ that contains X. With the help of theorem 1 it may be easily shown that the point set $K \cdot D_O$ contains an uncountable subset L such that if X and Y are any two points of L then N_X and N_Y have a point in common. Let P denote any given point of L. The domain I_P contains N_X for every point X of L. Thus the sum N of all the sets N_X for all points X of L is a connected subset of I_P. Furthermore, N contains P. Hence, it is a subset of M_P. Thus, for every point P of L, $N + N'$ contains N_P and is a subset of M_P. It follows, by lemma 2, that every point of L is a cut point of $N + N'$. Furthermore, no point of L is a point, of Menger order two, of $N + N'$. But, by a theorem of G. T. Whyburn's,[13] no continuum has uncountably many cut points that are not of Menger order two. Thus the supposition that theorem 5 is false has led to a contradiction.

[1] Menger, K., "Ueber reguläre Baumkurven," *Math. Ann.*, **96**, 1926 (572–582).

[2] Alexandroff, P., "Ueber kombinatorische Eigenschaften allgemeiner Kurven," *Math. Ann.*, **96**, 1926 (512–554).

[3] C. Kuratowski and C. Zarankiewicz, "A Theorem on Connected Point Sets," *Bull. Amer. Math. Soc.*, **33**, 1927 (571–575).

[4] Menger, K., "Zur allgemeiner Kurventheorie," *Fund. Math.*, **10**, 1927 (96–115).

[5] Wazewski, T., "Sur les courbes de Jordan ne renfermant aucune courbe simple fermee de Jordan," *Annales de la Société Polonaise de Mathématique*, **2**, 1923 (49–170).

[6] A continuous curve is said to be acyclic if it contains no simple closed curve. Cf. Gehman, H. M., "Concerning Acyclic Continuous Curves," *Trans. Amer. Math. Soc.*, **29**, 1927 (553–568).

[7] Janiszewski, S., "Sur les coupures du plan faites par les continus" (en polonais), *Prace matematyczno-fizyczne*, **26**, 1913 (11–63).

[8] Rosenthal, A., "Teilung der Ebene durch irreduzible Kontinua," *Münchener Akademie, Sitzungsber.*, **1919** (91–109).

[9] Mullikin, Anna M., "Certain Theorems Relating to Plane Connected Point Sets," *Trans. Amer. Math. Soc.*, **24**, 1922 (144–162).

[10] Knaster, B., "Un continu dont tout sous-continu est indécomposable," *Fund. Math.*, **3**, 1922 (247–286).

[11] Janiszewski and Kuratowski, "Sur les continus indécomposables," *Fund. Math.*, **1**, 1920 (210–222).

[12] A continuous curve is said to be simple if it is either an arc, an open curve, a simple closed curve or a ray of an open curve. See my paper "Concerning Simple Continuous Curves," *Trans. Amer. Math. Soc.*, **21**, 1920 (333–347).

[13] Whyburn, G. T., "Concerning the Cut Points of Continua." Presented to the American Mathematical Society, Sept. 9, 1927, but not, as yet, published in full.

ON METHODS AND APPLICATIONS IN SPECTROPHOTOMETRY

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1. The interpretation of stellar spectra is not possible without precise photometry. Just as it was necessary that stellar spectra should be photographed before they could be discriminatingly classified, it is essential that the photographed spectra be measured before they can be interpreted in the light of modern theory. The inadequacy of eye estimates for photographic photometry has repeatedly been demonstrated and, accordingly, recent spectroscopic work at Harvard has been largely concerned with the development and standardization of methods of photographic spectrophotometry by means of the Moll microphotometer. The present note is intended to summarize the results of two years' experiments.

The basic problem of the calibration of a photographic plate involves the determination of the blackening that corresponds to a given intensity of incident light. As the photographic plate responds selectively to light of different wave-lengths, and as the nature of this response varies for different emulsions, a complete standardization for any one emulsion