the flower colors in the resulting progenies, while in the second type of cross it is the *Ss* factors whose segregations determine the flower colors of the families produced. The *Ss* factors are in linkage group I in which the *rubricalyx* factor is also included, while the *Vv* factors are in linkage-group III. It is clear, therefore, that independence between old-gold and *rubricalyx* can be observed whenever the *Vv* factors alone are segregating and the *SS* factors remain constant; whereas, a close linkage may be observed between old-gold and *rubricalyx*, or other first-chromosome factors, when it is the *Ss* pair that is segregating and the *vv* factors remain constant.

[1] Read before the Joint Genetics Sections of the Botanical Society of America and the American Society of Zoölogists, at Nashville, December 29, 1927. The experiments on which this paper is based have been supported in part by grants from the American Association for the Advancement of Science, the Elizabeth Thompson Science Fund and the BACHE FUND OF THE NATIONAL ACADEMY OF SCIENCES.

[2] Shull, G. H., "'Old-Gold' Flower Color, the Second Case of Independent Inheritance in Oenothera," *Genetics*, **11**, 201–234 (1926).

A NEW PROOF OF THE LEFSCHETZ FORMULA ON INVARIANT POINTS

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Communicated January 9, 1928

1. The sum of the indices of the fixed points of a given transformation, which, since Brouwer's[1] first proofs of fixed point theorems, was the subject of many special investigations, has been completely determined for arbitrary transformations of arbitrary manifolds by Lefschetz.[2] His theory includes the fixed point formula as a special case of more general theorems on coincidences and multiply valued transformations, and he also makes the remark that it is possible to apply the same methods to certain transformations of an arbitrary complex.[3]

In the following there will be sketched a new proof of the fixed point formula. This proof holds for all complexes, under the assumption that the transformation is one-valued; the question whether it holds also for multiply-valued transformations will not be treated here. A paper with all details will be published in the *Mathematische Zeitschrift.*

2. *The Lefschetz formula.*—Let f be a one-valued continuous transformation of an n-complex C^n into itself and $\gamma^i_1, \gamma^i_2, \ldots, \gamma^i_{p^i}$ a fundamental set of i-cycles on C^n. Then there exists a system of homologies

$$f(\gamma_j^i) \sim \sum_{k=1}^{p^i} a_{jk}^i \gamma_k^i + \nu^i \quad (j = 1,2,\ldots, p^i), \tag{1}$$

where ν^i is a zero-divisor. One sees without difficulty[4] that the trace $\sum_{j=1}^{p^i} a_{jj}^i$ of this substitution does not depend on the choice of the set γ_j^i, but is a constant of f, to be called $S^i f$. Thus to the given transformation f there belong $n + 1$ constants $S^0 f, S^1 f, \ldots, S^n f$. Now the fixed points formula proved by Lefschetz for the case where C^n is a manifold[5] says that, if $\xi_1, \xi_2, \ldots, \xi_m$ are the invariant points of f and $j_1, j_2, \ldots, j_m$ their indices, then

$$\sum_{g=1}^{m} j_g = (-1)^n. \sum_{i=0}^{n} (-1)^i S^i f. \tag{2}$$

3. *A generalization of the Euler-Poincaré formula.*—In order to prove (2) we first consider an "elementary transformation"[6] φ of C^n into itself, which transforms the vertices of a subdivision C_1^n of C^n into the vertices of C^n. By φ each i-simplex T_j^i of C_1^n is transformed into an i-simplex $\varphi(T_j^i)$ of C^n, which may also degenerate to less than i dimensions. However, because each i-simplex of C^n is decomposed into i-simplices of C_1^n, we have a system of equations

$$\varphi(T_j^i) = \sum_{k=1}^{a^i} c_{jk}^i T_k^i \quad (j = 1, 2, \ldots, a^i), \tag{3}$$

where a^i is the number of i-simplices of C_1^n and where the c_{jk}^i are equal to ± 1 or to 0.

We say that between the traces of these square matrices $\| c_{jk}^i \|$ and the constants $S^i\varphi$ there holds the following relation:

$$\sum_{i=0}^{n} (-1)^i \sum_{j=1}^{a^i} c_{jj}^i = \sum_{i=0}^{n} (-1)^i S^i \varphi. \tag{4}$$

When φ is the identity, then the matrix $\| c_{jk}^i \|$ in (3) as well as the matrix $\| a_{jk}^i \|$ in (1) is the matrix unity. Therefore, in this case (4) is reduced to the Euler-Poincaré formula[7]

$$\sum_{i=0}^{n} (-1)^i a^i = \sum_{i=0}^{n} (-1)^i p^i. \tag{4*}$$

(4) can be proved by induction. It is obviously correct for $n = 0$. Assume it proved for any $(n-1)$-complex. Then it holds for the elementary transformation φ^1 of the complex C^{n-1}, formed by the $(n - 1)$-simplices of C^n, where φ^1 is identical with φ on C^{n-1}. So we have

$$\sum_{i=0}^{n-1} (-1)^i \sum_{j=1}^{a^i} c_{jj}^i = \sum_{i=0}^{n-1} (-1)^i S^i \varphi^1. \tag{4_{n-1}}$$

But we have also

$$S^i\varphi^1 = S^i\varphi \quad (i = 0, 1, \ldots, n-2) \tag{5a}$$

$$S^{n-1}\varphi^1 = S^{n-1}\varphi - S^n\varphi + \sum_{j=1}^{a^n} c^n_{jj}, \tag{5b}$$

of which (5*a*) are self-evident, while (5*b*) may be proved without great trouble. Replacing $S^i\varphi^1$ in (4_{n-1}) by the aid of (5*a*), (5*b*) formula (4) follows.

4. *The fixed point formula for transformations without fixed points.*—Let now f be any one-valued continuous transformation of C^n into itself, which possesses no fixed point, and φ an elementary transformation of C^n which approximates f sufficiently closely.[8] Then

$$S^i\varphi = S^i f \quad (i = 0, 1, \ldots, n) \tag{6}$$

$$c^i_{jj} = 0 \quad (i = 0, 1, \ldots, n;\ j = 1, 2, \ldots, a^i); \tag{7}$$

from these equations together with (4) there follows

$$\sum_{i=0}^{n} (-1)^i S^i f = 0. \tag{8}$$

This is the Lefschetz formula for an f without fixed points.

5. *A modification of a transformation in the neighborhood of its fixed points.*—If f has invariant points, then we shall confine ourselves to the case where the number of these points is finite and where each has an Euclidean neighborhood, so that we may assume that it is lying in the interior of an n-simplex of C^n. Let ξ be an invariant point, T^n an n-simplex containing ξ, and t^n another simplex containing ξ which is so small that its image $f(t^n)$ is also in the interior of T^n. Then take an $(n-1)$-sphere H^{n-1} in T^n and let to any point x of the bounding sphere r^{n-1} of t^n correspond the intersection point x' of H^{n-1} with the ray through the center of H^{n-1}, which is parallel to the vector $\overrightarrow{xf(x)}$. The Brouwer "degree"[1] of this representation of the sphere r^{n-1} on the sphere H^{n-1} is, by definition, the "index" j of ξ.

Let us call ξ a "normal" invariant point, if there exists a t^n, which has no point in common with the image $f(r^{n-1})$ of its boundary. Then it follows easily from fundamental properties of the degree, that j is the degree of the representation $f(t^n)$ in each point of the interior of t^n, i.e., that j for each such point is the algebraic number of coverings under a simplicial approximation of $f(t^n)$.

It may readily be shown that by slightly modifying f, ξ is turned into a normal fixed point. The modification leaves invariant the number of fixed points, their positions and indices, as well as the numbers $S^i f$. Therefore, we may properly assume henceforth that all fixed points are normal.

6. *Reduction of a transformation with invariant points to a transformation without invariant points.*—By the following construction which erplaces each invariant point by an n-cycle, transformed into itself, we reduce the proof of (2) to formula (8), proved above.

Let $\xi_1, \xi_2, \ldots, \xi_m$ be the fixed points, t_g^n ($g = 1, 2, \ldots, m$) the simplex, containing ξ_m and having the property described in No. 5, which defines the "normality" of ξ_g, further r_g^{n-1} the boundary of t_g^n. Then for each g we add to the complex C^n a new n-cell $\overline{t_g^n}$, which has the same boundary r_g^{n-1}, but, except the points of r_g^{n-1}, has no other point in common with C^n. Thus C^n has been enlarged to an n-complex $\overline{C^n}$, on which a fundamental set of n-cycles consists of a fundamental set on C^n together with m new n-spheres $\pi_g^n = t_g^n + \overline{t_g^n}$. The fundamental sets of the other dimensionalities have not been changed.

Let G be the one-one transformation of $\overline{C^n}$, which is the identity in all points not belonging to a π_g^n and which on each sphere π_g^n is the reflection with respect to the equatorial $(n-1)$-sphere r_g^{n-1}. Let further $\bar{f}$ be the one-valued and continuous transformation of $\overline{C^n}$ into itself defined in the following way:

$$\bar{f}(x) = Gf(x), \text{ if } x \subset C^n$$

$$\bar{f}(x) = fG(x), \text{ if } x \not\subset C^n$$

$\bar{f}$ has no invariant point, hence (8) gives here:

$$\sum_{i=0}^{n} (-1)^i S^i\bar{f} = 0. \tag{$\bar{8}$}$$

Now, from the defining property of the "normal" fixed points together with the fact that G has the degree -1 in each point of π_g^n, there follows that the share of π_g^n in the trace $S^n\bar{f}$ is equal to $-j_g$, where j_g is the index of ξ_g. The share of any i-cycle of C^n in S^if is unchanged when we replace f by $\bar{f}$. Therefore, we have

$$S^i\bar{f} = S^if \quad (i = 0, 1, \ldots, n-1) \tag{9}$$

$$S^n\bar{f} = S^nf - \sum_{g=1}^{m} j_g. \tag{10}$$

From (8), (9), (10) follows the formula

$$\sum_{g=1}^{m} j_g = (-1)^n \sum_{i=0}^{n} (-1)^i S^if. \tag{2}$$

[1] L. E. I. Brouwer, *Mathem. Ann.*, **71** (1911), pp. 97–115.

[2] S. Lefschetz (*a*) *Trans. Am. Math. Soc.*, **28** (1926), pp. 1–49; (*b*) **29** (1927), pp. 429–462.

[3] S. Lefschetz, *Proc. Nat. Acad. Sci.*, **13** (1927), pp. 621–622.

[4] See Lefschetz (*a*), No. 71.

[5] Lefschetz (*a*), formula 71.1; (*b*) formulas (10.5), (36.2). The reason for the fact that these formulas differ from our formula (2) by the factor $(-1)^n$ is an inessential difference in the definition of the "index."

[6] J. W. Alexander, *Trans. Amer. Math. Soc.*, **28** (1926), pp. 305–306.

[7] See, for instance, Alexander, l. c., p. 316, formula 10.6.

[8] Alexander, l. c., pp. 306–307.

NOTE ON PROJECTIVE COÖRDINATES

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Communicated January 7, 1928

In a recent paper in these Proceedings O. Veblen[1] has developed a theory of projective tensors which is based on the fact that associated with a general analytic transformation and a given point there is a unique linear fractional transformation. It is the purpose of this note to derive the associated transformation in what would seem a more direct and significant way.

Let the given transformation be, in the neighborhood of the point x_0, in question,

$$x^i - x^i_o = u^i(\bar{x} - \bar{x}_o) = (u^i_j)_o(\bar{x}^j - \bar{x}^j_o) + \frac{1}{2}(u^i_{jk})_o(\bar{x}^j - \bar{x}^j_o)(\bar{x}^k - \bar{x}^k_o) +. \quad (1)$$

The associated linear fractional transformation

$$y^i = \frac{a^i_j\bar{y}^j}{1 + b_k\bar{y}^k} = a^i_j\bar{y}^j - a^i_j b_k \bar{y}^j\bar{y}^k + \ldots \quad (2)$$

is determined by the two requirements:

(*a*) It shall agree with (1) in terms of 1st order. This condition alone associates a unique linear transformation with (1).

(*b*) Its Jacobian shall agree with that of (1) in terms of 1st order; i.e., the ratios of volume magnification shall differ by at most terms of 2nd order.

From (*a*) it follows that

$$a^i_j = (u^i_j)_o \quad (3)$$

The Jacobian of the original transformation is

$$u = \left| (u^i_j)_o + (u^i_{jk})_o(\bar{x}^k - \bar{x}^k_o) + \ldots \right| = u_o\left\{1 + (v^j_i u^i_{jk})_o(\bar{x}^k - \bar{x}^k_o) + \ldots\right\}$$

and that of (2) is