The simplicity of these fundamental postulates and the ease in which they can be applied and verified by experiment open up alluring possibilities for their application throughout the whole range of organic compounds and seem to present a most valuable clew for extending our knowledge of complex molecular structures.

Besides the intrinsic interest of the results herewith presented on acetylene this work offers a convenient point of departure for (1) the interpretation of the x-ray diffraction effects from solid acetylene and the determination of the lattice structure, (2) the study of the relative abundance of the various dynamic isomers through an analysis of the effect of temperature on the intensities of these bands, and (3) a study of the possible electron configurations in the acetylene molecule and the physical effects to be looked for from them. Since there are fourteen electron positions, and since there are only ten electrons available to fill them, the unsaturated character of acetylene is accounted for on the model, and different allocations of the available electrons in these possible electron positions allow for the occurrence of different electronic structures which may be called "electromers" of acetylene.

[1] J. K. Morse, "The Structure and Dimensions of the Ethane Molecule," *Proc. Nat. Acad. Sci.*, **14**, 37–40 (1928).

[2] J. K. Morse, "Atomic Lattices and Atomic Dimensions," *Proc. Nat. Acad. Sci.*, **13**, 227–32 (1927).

[3] J. K. Morse, "The Lattice Structure of Ethane," *Proc. Nat. Acad. Sci.*, **14**, 40–5 (1928).

[4] J. K. Morse, "The Structures of Methane," *Proc. Nat. Acad. Sci.*, **14**, 166–71 (1928).

[5] A. Levin and C. F. Meyer, "The Infra Red Absorption Spectra of Acetylene, Ethylene and Ethane," *J. Optical Soc. Am.*, **16**, 137–64 (1928).

ON DERIVATIVES OF NON-ANALYTIC FUNCTIONS

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Communicated July 2, 1928

1. *Introduction.*—It is my purpose in this paper to discuss the relations that exist between the recent papers by Kasner[1] and the paper by Ingold, Westfall, and myself,[2] regarding the derivatives of functions of a complex variable which are not analytic. These we have called *non-analytic;* Kasner has called them *polygenic,* but he also uses the name non-analytic. Some of the ideas of these papers evidently agree, but there are some which do not agree on the face of things. At one point, indeed, I shall introduce a notation different from that of our former paper, in order to make clear

the intimate connection that exists, but I shall point out that these changes do not affect the essential character of the concepts discussed by us previously.

Kasner has (loc. cit.) considered the values of the limit of the increment-ratio

$$\gamma = \alpha + i\beta = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \qquad (1)$$

of a non-analytic (or polygenic) function $w = u + iv = f(z)$ of the complex variable $z = x + yi$. Using his notation, $u = \phi(x, y)$, $v = \psi(x, y)$, we have

$$\gamma = \alpha + i\beta = \frac{\phi_x + i\psi_x + m(\phi_y + i\psi_y)}{1 + im} \qquad (2)$$

where m denotes the slope dy/dx of the path along which Δz approaches zero, and where γ depends in general on m. He shows that the locus of γ for various values of m is a circle:

$$\left(\alpha - \frac{\phi_x + \psi_y}{2}\right)^2 + \left(\beta - \frac{-\phi_y + \psi_x}{2}\right)^2 = \left(\frac{\phi_x - \psi_y}{2}\right)^2 + \left(\frac{\phi_y + \psi_x}{2}\right)^2, \qquad (3)$$

whose radius is zero if and only if the Cauchy-Riemann equations

$$\left.\begin{aligned} \phi_x &= \psi_y, \\ \phi^y &= -\psi_x \end{aligned}\right\} \qquad (4)$$

hold at the point (x, y). This circle Kasner calls the *derivative circle.* I shall call it the *Kasner circle.*

2. *Diameters of the Kasner Circle.*—Kasner has used the fact that the point

$$\gamma = \alpha + i\beta = \psi_y + i\psi_x$$

lies on the circle (3), and has given a simple construction for determining the point on the circle that corresponds to any slope m. It is obvious that the points that correspond to $m = 0$ and to $m = \infty$, that is,

$$\frac{dw}{dx} = \phi_x + i\psi_x, \text{ and } \frac{dw}{dy} = \psi_y - i\phi_y \qquad (5)$$

also lie on the Kasner circle, and that they are at the ends of a diameter of it. These points give an easy construction, and they furnish a good mnemonic for the center and the radius of the circle.

Another diameter of the circle is furnished by finding the maximum and the minimum distances from the origin in the (α, β) plane to the circle. This we did in our paper (loc. cit.). We set

$$r = \alpha^2 + \beta^2 = \left|\frac{dw}{dz}\right|^2 = \frac{du^2 + dv^2}{dx^2 + dy^2}, \qquad (6)$$

from which, after easy cancellations, we found

$$r = \left| \frac{dw}{dz} \right|^2 = \frac{E + m^2(E + G') + m^4G + 2m(1 + m^2)F}{(1 + m^2)^2} = \frac{E + 2mF + Gm^2}{1 + m^2}, \quad (7)$$

where E, F and G are defined by the usual equations

$$E = \phi_x^2 + \psi_x^2, \qquad G = \phi_y^2 + \psi_y^2, \qquad F = \phi_x\phi_y + \psi_x\psi_y.$$

We found that the maximum and the minimum of r are given by the equation

$$F + (G - E)m - Fm^2 = 0, \quad (8)$$

and that they are the solutions of the quadratic*

$$\rho^2 - (E + G)\,\rho + J^2 = 0. \quad (9)$$

3. *The Indicatrix.*—In our paper, we used the ratio r, rather than its square root, for convenience; and we defined two concepts, each of which would not be affected materially by such a change. Calling the solutions of the equation (9) ρ_1 and ρ_2, we called the ellipse whose semi-axes are ρ_1 and ρ_2, the *indicatrix*, and we called the difference

$$|\rho_1 - \rho_2| \quad (10)$$

the *ellipticity*, as a measure of the departure of the ellipse from circularity. Since we were concerned chiefly with the directions of the axes, which are the solutions of (8), we chose the ellipse and the measure of ellipticity in as simple a manner as possible. Any other simple measure based on ρ_1 and ρ_2 would have served our purpose as well; for example, the difference

$$d = \sqrt{\rho_1} - \sqrt{\rho_2} \quad (11)$$

is an equally satisfactory measure. *This difference d is precisely the diameter of the Kasner circle.* I propose henceforth to call d the *ellipticity*.

A similar change of notation is desirable in the case of the indicatrix, in order to bring it into close relation with the ideas discussed above. As we defined it, the directions were the solutions of (8), and the semi-axes were ρ_1 and ρ_2. There exists another ellipse with the same principal directions, whose radial distances bear closer relations to the preceding concepts. For, let us plot as polar distances from the original values (x, y), the *stretching factor* P defined by the equations

$$P^2 = \frac{1}{r} = \frac{ds^2}{d\sigma^2} = \left| \frac{dz}{dw} \right|^2 = \frac{dx^2 + dy^2}{du^2 + dv^2}, \quad (12)$$

where s and σ are the arc-lengths in the (x, y) and the (u, v) planes, respectively. Let us set $\bar{x} = x - x_0$, $\bar{y} = y - y_0$, and lay off P in the direction $m = dy/dx$. Then we shall have

$$P^2 = \bar{x}^2 + \bar{y}^2, m = \bar{y}/\bar{x},$$

and (12) becomes

$$E\bar{x}^2 + 2F\bar{x}\,\bar{y} + G\bar{y}^2 = 1, \tag{13}$$

which is an ellipse, since

$$J^2 = \begin{vmatrix} \phi_x\phi_y \\ \psi_x\psi_y \end{vmatrix}^2 = H^2 = EG - F^2 > 0, \tag{14}$$

provided, of course, that J is not zero at (x_0, y_0).

Likewise, if J is not zero at (x_0, y_0), and with mild assumptions of continuity, we can solve for x and y in terms of u and v near (x_0, y_0), and we may write

$$x = \Phi(u, v), \quad y = \Psi(u,v), \tag{15}$$

$$\mathfrak{E} = \Phi_u^2 + \Psi_u^2, \ \mathfrak{F} = \Phi_v^2 + \Psi_v^2, \ \mathfrak{G} = \Phi_u\Phi_v + \Psi_u\Psi_v, \tag{16}$$

$$\mathfrak{J}^2 = \mathfrak{E}\mathfrak{F} - \mathfrak{G}^2 = \frac{1}{EG - F^2} = \frac{1}{J^2}. \tag{17}$$

With this notation, we may draw, in the (u, v) plane, about the point (u_0, v_0), an ellipse whose polar distances are the values of R, where

$$R^2 = r = \frac{d\sigma^2}{ds^2} = \left|\frac{dw}{dz}\right|^2 = \frac{du^2 + dv^2}{dx^2 + dy^2} = \frac{du^2 + dv^2}{\mathfrak{E}du^2 + 2\mathfrak{F}dudv + \mathfrak{G}dv^2}. \tag{18}$$

The equation of this ellipse, in terms of $\bar{u} = u - u_0$, and $\bar{v} = v - v_0$, is

$$\mathfrak{E}\bar{u}^2 + 2\mathfrak{F}\bar{u}\,\bar{v} + \mathfrak{G}\bar{v}^2 = 1. \tag{19}$$

The ellipse (19) has as its radial distances precisely the distances from the origin in the (α, β) plane to the corresponding points of the Kasner circle, and the difference between its semi-axes is the diameter of the Kasner circle. This ellipse is, in fact, the polar diagram of the Kasner circle. The difference of its semi-axes, in the older notation, is

$$\sqrt{\rho_1} - \sqrt{\rho_2},$$

which is the diameter of the Kasner circle. It is evident that the changes made above are not material from the standpoint of the earlier paper, since the concepts discussed there, particularly the concepts called *principal directions, characteristic lines, analytic at a point, functions of the same class*, etc., would be the same under either set of notations. For the purpose of displaying the relationships mentioned above, however, it is advantageous to use the stretching ratios R and P defined above, which are expressed in terms of the original ratio r by the simple formulas

$$R^2 = r, \quad P^2 = \frac{1}{r};$$

and it is advantageous to substitute the ellipses (13) and (18) in the (x, y) and the (u, v) planes, respectively, for the somewhat arbitrary ellipse used in the original paper.

4. *The Jacobian of the Increment-Ratio.*—The increment-ratio itself,

$$\zeta = \xi + i\,\eta = \frac{\Delta w}{\Delta z} = \frac{w - w_0}{z - z_0} = \frac{f(z) - f(z_0)}{z - z_0} = F(z), \qquad (20)$$

defines a function $F(z)$ of the complex variable z, except for $z = z_0$. For an analytic function, or for a non-analytic one that has an isolated analytic point at $z = z_0$, the derivative exists uniquely, and its value completes in a continuous manner the definition of $F(z)$ at $z = z_0$. If this derivative vanishes, however, there are in general two leaves of a Riemann surface which connect together about the point $F(z_0)$, and the rate of angular turn about that point is in general doubled.

When $f(z)$ is not analytic at $z = z_0$, some (but not all) of the properties that depend upon the derivative in the analytic case can be stated in terms of the jacobian

$$J = \frac{D(\xi, \eta)}{D(x, y)} = \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix}.$$

It is not difficult to calculate this jacobian, but I shall omit it here for brevity. It turns out, very simply, that the locus of the points at which the jacobian vanishes, that is, the locus of the equation

$$J = \frac{D(\xi, \eta)}{D(x, y)} = 0,$$

is precisely the Kasner circle in the (ξ, η) plane, which coincides with the (α, β) plane of Kasner's paper insofar as these points are concerned. I may therefore state the theorem that follows.

THEOREM. *The equation of the Kasner circle is $J = 0$.*

In general, the vanishing of the jacobian for any non-analytic function (that is, for any point transformation of the plane) defines a curve of which the branch-point of an analytic function is a degenerate case, since the jacobian for an analytic function vanishes if and only if the derivative vanishes. For analytic functions, this can happen only at isolated points, since, by the Cauchy-Riemann equations, the jacobian reduces to a sum of squares for an analytic function. For a non-analytic function, however, the locus of the points at which the jacobian vanishes is in general a curve.

The corresponding curve in the (u, v) plane may be called the *edge of regression*, on account of its similarity to the edge of regression of a developable surface. Indeed, the Riemann surface for such a function resembles decidedly a developable surface, since curves in the z plane through any point of it correspond to curves that are tangent to the corresponding curve in the (u, v) plane.

The edge of regression of the increment-ratio $\zeta = \Delta w/\Delta z$ is then precisely the Kasner circle. It is the envelope of curves corresponding to any pencil of curves through the point $z = z_0$ in the (x, y) plane. The Riemann surface over the ζ plane is in general two-leaved, joining along the points of the Kasner circle. The fact discovered by Kasner that the rate of rotation on the Kasner circle is double the rate in the (x, y) plane is a generalization of the fact that the rate of rotation for an analytic function about a branch-point is double the rate about the corresponding critical point.

I shall discuss these concepts in more detail elsewhere.

* There is a misprint in this equation in our original paper. It occurs as equation (10) on p. 335; in that equation the constant term should be J^2.

[1] See E. Kasner, *Science*, **66**, 581–582 (1927); and these PROCEEDINGS, **14**, 75–82 (1928).

[2] See Hedrick, Ingold and Westfall, "Theory of Non-Analytic Functions of a Complex Variable," *J. Math.*, Ser. (9), **2**, 327–342 (1923).

RADIATION AND RELATIVITY. II[1]

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Communicated June 28, 1928

1. In the first paper[2] we arrived at an electromagnetic field which—we are led to believe—is likely to accompany a light corpuscle, and we found that it possesses some properties we are accustomed to associate with light, but it is not periodical. From an entirely different point of view we can arrive at a field in which periodicity appears, so to say, automatically, and although it is not clear at present whether the two points of view can be combined successfully it is probably not without interest to take up this second point of view.

2. It has been proved several years ago[3] that in general relativity theory when the curvature tensor is given the electromagnetic tensor is determined (up to a constant) and that the expressions for the electromagnetic tensor involve trigonometric functions. It was concluded from this that the electromagnetic field *might* be periodic even when the curvature field is not. But an example of such a situation was lacking; we cannot give an ex-