[2] Department of Zoölogy, Contribution No. 221.

[3] Since the above was written Dr. Robertson has shown me figures (unpublished) in which a longitudinally split condition of the chromosomes in the spermatids of grasshoppers is evident, and Dr. M. T. Harmon has informed me that she has observed the same phenomenon in the spermatids of guinea pigs. Various cytologists have observed a split condition of chromosomes in the telophase stages of other cell divisions. According to observations of Shiwago, chromosomes are double at all stages, and split secondarily just before the identical halves separate in mitosis.

[4] This paragraph has been inserted since the reading of the original paper, as the reversion was not observed until just after the author's return from the meeting. In the original paper it had been stated that no reversions of forked had been obtained. The writer wishes to thank Professor J. T. Patterson for invaluable aid in this experiment, rendered during the writer's absence on the trip.

[5] E. Stein (1922, 1926) had previously reported numerous phenotypic abnormalities of Antirrhinum, produced by radiation, which were inherited through vegetative reproduction. She also noted abnormalities of chromosome distribution occurring at the reduction division, long after treatment. In animals, non-disjunction of the X and other chromosomes at maturation, produced by radiation, was first demonstrated by Mohr (1919) in *Decticus*.

ON THE ROOTS OF THE DERIVATIVE OF A POLYNOMIAL

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The purpose of this note is to indicate the nature of some results obtained by the writer in generalizing a group of theorems proved several years ago by Professor Walsh of Harvard University concerning the approximate geometric location of the roots of the derivative of a polynomial.[2] A principal result is the following

Theorem. Suppose m_i roots of a polynomial $f(z)$ of degree n have as their common locus a region C_i consisting of the interior and the circumference of the circle

$$C_i = x^2 + y^2 - 2\alpha_i x - 2\beta_i y + \gamma_i = 0,$$

where $i = 1, 2, \ldots, q + 1,$ *and* $\sum_{i=1}^{q+1} m_i = n.$

Then the roots of the derivative of $f(z)$ have as their locus

(*a*) *the common points of any two of the regions C_i;*

(*b*) *the points of every region C_i for which $m_i \neq 1$; and*

(*c*) *the interior and the boundary of all the ovals of the q-circular 2q-ic curve*

$$\sum_{i=1}^{q+1} \frac{nm_i}{C_i} - \sum_{\substack{i=1\\ j=i+1}}^{q+1} \frac{m_i m_j \sigma_{ij}}{C_i C_j} = 0, \qquad (1)$$

where σ_{ij} is the square of the common external tangent of the circles C_i and C_j.[3]

The singular foci of (1) *lie at the roots of the logarithmic derivative of*

$$g(z) = \prod_{j=1}^{q+1} (z - \delta_j)^{m_j},$$

where $\delta_j = \alpha_j + i\beta_j$.

If no two of the regions (*c*), $C_1, C_2, \ldots, C_{q+1}$ *have any point in common, they contain respectively* $q, m_1 - 1, m_2 - 1, \ldots, m_{q+1} - 1$ *roots of* $f'(z)$. *If, in addition,* (*c*) *falls into q distinct parts, each part contains just one root of* $f'(z)$.

This theorem may be proved by a method of "iterated envelopes."[4] First, only the m_1 roots of $f(z)$ within C_1 are permitted to vary, all the remaining roots of $f(z)$ being held fast, and the locus R_1 of the roots of $f'(z)$ under these conditions is found. Then the envelope R_2 of the R_1 when, in addition, the m_2 roots of $f(z)$ within C_2 are allowed to vary is obtained; and so on. Finally, by mathematical induction, the envelope R_{N+1} of the R_N when the m_{N+1} roots of $f(z)$ within C_{N+1} are also made to vary is secured. The proof of the theorem is completed by setting $N = q$.

The theorem has been verified in Professor Walsh's special cases, and a similar theorem obtained for the case that some or all of the regions C_i are the exteriors of circles. By essentially the same methods, some of Professor Walsh's results in connection with the location of the roots of the Jacobian of two binary forms and of the derivative of a rational function[5] have also been generalized analytically to any number of circular regions C_i.

The writer hopes presently to be able to report upon similar problems involving higher derivatives of a polynomial and linear combinations of these derivatives.

[1] NATIONAL RESEARCH FELLOW.

[2] Walsh, J. L., *Comptes Rendus du Congrès International des Mathématiciens*, Strasbourg, 1920; these PROCEEDINGS, **8** (1922), 139–141; *Trans. Amer. Math. Soc.*, **24** (1922), 52.

[3] In case the two circles C_i and C_j do not have a common external tangent, we shall use the expression

$$\sigma_{ij} = d_{ij}^2 - (r_i - r_j)^2,$$

where r_i and r_j are the radii of C_i and C_j, respectively, and d_{ij} is the distance between the centers of C_i and C_j.

[4] Cf. Walsh, J. L., these PROCEEDINGS, **8** (1922), p. 140, and *Trans. Amer. Math. Soc.*, **22** (1921), p. 102, footnote.

[5] Walsh, J. L., *Trans. Amer. Math. Soc.*, **19** (1918), 291–8; **22** (1921), 101–16; and **24** (1922), 31–69.