LINEAR TRANSFORMATIONS IN HILBERT SPACE. I. GEOMETRICAL ASPECTS[1]

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Communicated January 17, 1929

In this and succeeding notes, we summarize and extend some recent developments of Hilbert's classical theory of transformations in space of infinitely many dimensions. Any thorough discussion of the current formulation of the quantum theory depends upon the theory to be described.[2] Notable results in the direction of a complete treatment have been established by J. von Neumann, who has dealt with transformations in real space.[3] Since the quantum theory involves transformations of complex space, a more extended discussion appears to be desirable. The methods of von Neumann seem to be incapable of application to the more general problem, and cannot in any case be considered as the most powerful or satisfactory for the end in view. By a suitable combination of fundamental methods introduced by Carleman[4] and ideas elaborated by von Neumann, we are able to deal successfully with transformations in complex space. The advance is made possible by the introduction and utilization, ab initio, of the inverses of certain transformations. The simplification of analytical details brought about in this way can hardly be over-emphasized.

A complex abstract Hilbert space $\mathfrak{H}$ is a set of elements, denoted hereafter by f, g, . . ., which is characterized by five properties: linearity, the existence of a unitary metric, the non-existence of a finite basis, completeness, and separability. The metric is determined by a numerically valued function, $Q(f,g)$, defined for all pairs of elements of $\mathfrak{H}$; this function is an Hermitian symmetric bilinear function of its arguments and has the property that $Q(f) = Q(f,f)$ is real and never negative, vanishing if and only if f is the null element; the square of the distance between the elements f and g is $Q(f - g)$.

A transformation or operator T is a correspondence between two not-empty subsets, $\mathfrak{F}$ and $\mathfrak{D}$, of $\mathfrak{H}$ such that to each element of $\mathfrak{F}$ corresponds one element of $\mathfrak{D}$ and that each element of $\mathfrak{D}$ is the correspondent of at least one element of $\mathfrak{F}$. The sets $\mathfrak{F}$ and $\mathfrak{D}$ will be called the field and domain of T, respectively. If to each element of the field corresponds an element of the domain which is the correspondent of no other element of the field, the relation between the two sets is reciprocal and defines a transformation R which is called the inverse of T.

We consider only linear transformations with fields everywhere dense in $\mathfrak{H}$; a transformation T is said to be linear if its field is a linear set and

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if $T(a_1f_1 + a_2f_2) = a_1Tf_1 + a_2Tf_2$ for all complex numbers a_1 and a_2 and all elements f_1 and f_2 of its field. The linear transformations which satisfy the condition of boundedness, $Q(Tf) \leq C^2Q(f)$ with C a positive constant independent of f, are of considerable importance since they are handled with particular ease; a bounded linear transformation whose field is everywhere dense in $\mathfrak{H}$ may be extended by continuity to the entire space. Associated with a given linear transformation T is a unique linear transformation T^*, the adjoint of T; if there exist elements g and g^* such that $Q(Tf,g) = Q(f,g^*)$ for all f in the field of T, then g is in the field of T^* and $T^*g = g^*$, and conversely. If the field of T^* is everywhere dense in $\mathfrak{H}$, the adjoint of T^* is defined and is either identical with T or is an extension of T. We shall consider mainly two particular types of linear transformation: the self-adjoint transformations T, for which T and T^* are identical; and the unitary transformations U, for which U and U^* are defined throughout $\mathfrak{H}$ and are inverses of one another. As special cases we mention the following: the null-transformation or operator O, which takes every element of $\mathfrak{H}$ into the null element; the identity I, which takes every element of $\mathfrak{H}$ into itself; and the projection or special operator E which projects every element of $\mathfrak{H}$ onto a linear subspace, so that $E^2f = Ef$.

The fundamental problem of the theory of linear transformations is to prove the existence of linear subspaces of $\mathfrak{H}$ invariant under a given transformation T and to determine these subspaces. We consider here the first part of the solution of this problem for self-adjoint transformations.

If T is a self-adjoint transformation, the transformation $T_l \equiv T - l\cdot I$, where l is a complex parameter, is easily shown to have a unique bounded linear inverse R_l whose field is $\mathfrak{H}$, if l is not real. More generally, the following theorem can be asserted:

THEOREM.—A set of necessary and sufficient conditions that a family of linear transformations X_l defined over $\mathfrak{H}$ for all not-real l coincide with the family of inverses, R_l, of some family of transformations $T_l \equiv T - l\cdot I$, where T is a self-adjoint transformation, is the following:

(1) X_l is the adjoint of X_m, if l and m are conjugate complex numbers;

(2) $(l - m)X_lX_m = X_l - X_m$, for all l and m;

(3) there is at least one value of l such that $X_lf = 0$ implies that $f = 0$.

The proof is geometrical in character. By similar methods, a more refined result can be established, to wit:

THEOREM.—If T is a self-adjoint transformation, the points of the l-plane fall into three mutually exclusive sets A, B, C, with the following properties:

(1) A is an open set containing all not-real points of the l-plane and, possibly, some real points; if l is in A, then T_l has a unique bounded inverse R_l with field $\mathfrak{H}$, and, if l is also real, R_l is self-adjoint;

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(2) B, unless it is empty, contains only real points; if l is in B, T_l has a unique inverse R_l which is an unbounded self-adjoint transformation with a proper subset of $\mathfrak{H}$ as its field;

(3) C, unless it is empty, is a finite or denumerably infinite set of real points; if l is in C, then there is at least one element of the field of T for which $T_l f = 0$ while $Q(f) > 0$.

There seems to be no simple geometrical argument which will yield a demonstration of the fact that the set $B + C$ is not empty; even in n-dimensional space an appeal to the fundamental theorem of algebra is necessary to avoid difficult proofs. In the present discussion we will take up the question at a later stage.

[1] Presented to the American Mathematical Society, October 31, 1928.

[2] H. Weyl, *Zeit. Physik*, **46** (1927–28), pp. 1–46, and Gruppentheorie und Quantenmechanik, Leipzig, 1928; J. von Neumann, *Göttinger Nachrichten*, 1927, pp. 1–57 and 245–272.

[3] J. von Neumann, *loci citati;* and an unpublished paper which is to appear in the *Mathematische Annalen*, cf. *Göttinger Nachrichten*, 1927, pp. 1–55, footnotes 12 and 27.

[4] T. Carleman, *Equations intégrales singulières*, Uppsala, 1923.

ON COMMUTATION RULES IN THE ALGEBRA OF QUANTUM MECHANICS

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Communicated February 7, 1929

The algebra of the new quantum mechanics is a special form of the algebra of matrices. For one pair of canonically conjugate variables its properties are determined by the assumption

$$pq - qp = c, \tag{1}$$

where q and p are matrices which represent the coördinate and momentum, respectively. The value of the constant c plays no part in the development of the algebra although in the quantum mechanics it is assigned the value $h/2\pi i$.

The theory of differentiation of functions of a matrix has been discussed by Heisenberg, Born and Jordan.[1] Let f be a function of the argument matrix x. The definition of derivative used in this paper is

$$\frac{df(x)}{dx} = \lim_{\alpha \to 0} \frac{1}{\alpha} [f(x + \alpha \cdot 1) - f(x)].$$

Partial derivatives are defined in a similar manner. Born proved by

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