able that both are correct, but this cannot be certain until a sufficient fitter technic has been developed.

[1] Raman and Krishnan, *Nature* **121,** 501, 619, 711; **122,** 12, 169, 278 (1928). Landsberg and Mandelstam, *Naturwiss.*, **16,** 557 (1928).

[2] Pringsheim and Rosen, *Zeit. Physik,* **50,** 741 (1928).

[3] Wood, *Phil. Mag.*, (7) **6,** 729 (1928).

[4] Carelli, Pringsheim and Rosen, *Zeit. Physik,* **51,** 511 (1928).

LINEAR TRANSFORMATIONS IN HILBERT SPACE: II. ANALYTICAL ASPECTS[1]

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In our first note,[2] we introduced the terminology adapted to a study of linear transformations in complex Hilbert space and discussed certain geometrical aspects of self-adjoint transformations. We shall now develop analytical representations of such transformations or operators. For this purpose it is necessary to define the concept of a canonical resolution of the identity:[3] a canonical resolution of the identity is a family of special operators or projections, E_λ, defined for all real values of λ, with the properties

$$E_\lambda\, E_\mu = E_\mu\, E_\lambda = E_\mu, \qquad \lambda \geqslant \mu,$$
$$\lim_{\lambda \to -\infty} E_\lambda = O, \qquad \lim_{\lambda \to +\infty} E_\lambda = I,$$
$$\lim_{\epsilon \to 0} E_{\lambda+\epsilon} = E_\lambda, \qquad \epsilon > 0;$$

the limiting equations are to be interpreted in the sense that each is true when applied to an arbitrary element of $\mathfrak{H}$.

Our first analytical result is comprised in the following theorem.

THEOREM. Between the class of all self-adjoint transformations and the class of all canonical resolutions of the identity, there is a one-to-one correspondence such that, if R_l is the inverse of $T - l.I$, then

$$Q(R_l f, g) = \int_{-\infty}^{+\infty} \frac{1}{\lambda - l}\, dQ(E_\lambda f, g)$$

for all not-real l and for all f and g in $\mathfrak{H}$. The integral is to be taken in the sense of Stieltjes.

The proof of this theorem depends upon (1) the first theorem of our preceding note, (2) a uniqueness theorem due to Stieltjes, (3) a convergence theorem of Vitali, (4) the Helly-Bray theorem, (5) a theorem of

Fréchet concerning numerically-valued linear functions of an element in $\mathfrak{H}$; the method has been used by Carleman to obtain restricted conclusions of the same general character.[4] When T is given, we construct a sequence of self-adjoint transformations $T^{(N)}$ approximating T, in such a manner than $R_l^{(N)}$ can be represented as required by the theorem. By (3) we can then verify the equation $\lim_{N \to \infty} Q(R_l^{(N)} f, g) = Q(R_l f, g)$ for all not-real l. By (2) and (4) it is established that

$$Q(R_l f, g) = \int_{-\infty}^{+\infty} \frac{1}{\lambda - l}\, d\rho$$

where ρ is a unique function of limited variation, depending on f and g, with the properties

$$\lim_{\lambda \to -\infty} \rho(\lambda) = 0, \qquad \lim_{\lambda \to +\infty} \rho(\lambda) = Q(f, g), \qquad \rho(\lambda + 0) = \rho(\lambda).$$

Finally, we use (1), (2) and (5) to prove that $\rho = Q(E_\lambda f, g)$. When E_λ is given we can show without difficulty by (1), (2) and (5) that the transformation X_l defined by the relation $Q(X_l f, g) = \int_{-\infty}^{+\infty} \frac{1}{\lambda - l}\, dQ(E_\lambda f, g)$ is the inverse of a transformation $T - l.I$ for all not real l. We may remark that when T is known the family E_λ can be constructed from R_l by integration in the complex l-plane but that the validity of the procedure must be verified on the basis of the present theorem.[5]

By simple manipulations of the formula of this theorem we can derive the following direct relation between T and E_λ:

THEOREM. If T is a self-adjoint transformation and E_λ the corresponding canonical resolution of the identity, then Tf exists when and only when $\int_{-\infty}^{+\infty} \lambda^2 dQ(E_\lambda f)$ is finite, and $Q(Tf, g) = \int_{-\infty}^{+\infty} \lambda\, dQ(E_\lambda f, g)$ for all f in $\mathfrak{F}$ and all g in $\mathfrak{H}$.[6]

The two theorems enunciated in this note can now be combined with the second theorem of the preceding to complete our knowledge of R_l for all points of the l-plane.

THEOREM. (1) The real points of class A constitute an open set, on every open interval of which E_λ is constant; if l is a real point of class A, the analytic representation of R_l given above is valid for all f and g in $\mathfrak{H}$. (2) If l is a point of class B, then $E_{l-0} = E_l = E_{l+0}$, and E_λ is constant in no open interval containing l; $R_l f$ exists when and only when $\int_{-\infty}^{+\infty} \frac{1}{(\lambda - l)^2}\, dQ(E_\lambda f)$ is finite and is then representable in the form given above. (3) If l is a point of class C, then $E_{l-0} \neq E_l$ and there exists an $f \neq 0$ such that $Tf - lf = 0$; R_l is not defined for such a value of l. (4) The set $B + C$ is closed and not empty.

The terms spectrum, continuous spectrum and point spectrum customarily applied to the sets $B + C$, B and C, respectively, derive their significance from this theorem.

A partial solution of the problem of the existence of linear subspaces of $\mathfrak{H}$ invariant under T may now be formulated:

THEOREM. If (λ,μ) is a closed interval to which the point $\lambda = 0$ is exterior, and if $\mathfrak{H}_{\lambda,\mu}$ is the closed linear subspace of $\mathfrak{H}$ comprising all elements f such that $E_\lambda f - E_\mu f = f$, then T is defined throughout $\mathfrak{H}_{\lambda,\mu}$ and transforms it in a one-to-one manner into itself. The space $\mathfrak{H}_{\lambda,\mu}$ may contain the sole element 0, may be an n-dimensional unitary space, or may be a complex Hilbert space.

[1] Presented to the American Mathematical Society, October 27, 1928.

[2] *Proc. Nat. Acad. Sci.*, **15**, p. 198, 1929.

[3] J. von Neumann, *Göttinger Nachrichten*, **1927**, pp. 32–33. The appellation "resolution of the identity" seems to be an appropriate translation of his "Zerlegung der Einheit" and is descriptive of the properties of the family E_λ.

[4] Stieltjes, *Annales Toulouse*, 1, **8** (1894), pp. J68–J75; Montel, *Leçons sur les familles normales*, Paris, **1927**, p. 30; Bray, *Ann. Math.*, **20** (1918–19), pp. 177–186; Fréchet, *Trans. Am. Math. Soc.*, **8** (1907), pp. 439–44; Carleman, *Equations intègrales singulières*, Uppsala, **1923**, pp. 25–28, 52–55, 81–82.

[5] Hellinger, *J. Mathematik*, **136** (1909), pp. 210–271, employed the method of contour integration to discuss bounded operators.

[6] This relation has been proved recently by von Neumann in an unpublished paper, kindly brought to my attention by Professor Weyl since the preparation of my first note. His proof, also indirect, is entirely different from that sketched here. The other theorems I have given can be derived from this, once it has been demonstrated.

AGE CHANGES IN ALCOHOL TOLERANCE IN DROSOPHILA MELANOGASTER

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While certain orderly physiological changes with advancing age are known for some organisms, notably man, in general it is true that the stigmata of senescence which are best known are morphological. There has been relatively little work of a systematic character regarding senescence on any other organism than man.

It is the purpose of this paper to report in a preliminary way some of the results of a study of physiological changes with age in *Drosophila* upon which we have been engaged for some years past in this Institute. Later a more complete account of the investigation will be published in our series of *Experimental Studies on the Duration of Life*.