AN EXTENSION OF THE ALEXANDER DUALITY THEOREM

BY ARTHUR B. BROWN[1]

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

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1. *Removal of a Closed Set.*—Notations are as in the preceding paper.[2] If K is closed $R_i(K)$ is called the Vietoris Betti number if it is under the Vietoris definition,[3] as modified by Lefschetz[4] so as to apply to absolute Betti numbers. Unless otherwise stated, $R_i(K)$ is defined topologically. All formulas hold for $i = 0, 1, \ldots, n - 1$, unless otherwise stated.

THEOREM 1. *Let S be an n-manifold with the Betti numbers of an n-sphere. Let D be a point set on S with finite or infinite topological Betti numbers, and K a closed point set on a part of D which is open with respect to S. Then if D does not cover all of S, the following relations hold, where $R_{n-i-1}(K)$ is the Vietoris Betti number.*

$$R_i(D - K) = R_i(D) + R_{n-i-1}(K), \; i = 0, 1, \ldots, n - 1. \quad (1.1)$$

Proof.—By subdividing S we can obtain a sub-complex A which is an open n-manifold with boundary C an $(n - 1)$-manifold, such that K is interior to A and A is in the open part of D mentioned above. Let B and B_1 be the loci obtained from D and S, respectively, by removing the interior of A. We subdivide a part of B near C into cells, by taking cells of a subdivision of S. Now we apply (8.2) of the preceding paper, for A, B_1, C and S, and with $R_i(S)$ replaced by its value δ_{i0} we have

$$R_i(A) + R_i(B_1) = R_i(C) + \delta^i_{\,0} - \delta^i_{\,n-1}. \quad (1.2)$$

Since (1.1) are obviously satisfied if K is vacuous, we may assume that K is not vacuous. Then no one of $S - K$, D or $D - K$ covers S, and the following three applications of (8.1) of the preceding paper are valid.

$$R_i(A) + R_i(B) = R_i(C) + R_i(D); \quad (1.3)$$

$$R_i(A - K) + R_i(B_1) = R_i(C) + R_i(S - K); \quad (1.4)$$

$$R_i(A - K) + R_i(B) = R_i(C) + R_i(D - K). \quad (1.5)$$

Now we write the Alexander duality relation[5] as extended by Alexandroff,[6] Lefschetz and Alexander.[7,4,8]

$$R_i(S - K) = R_{n-i-1}(K) + \delta_{i0} - \delta^i_{\,n-1}. \quad (1.6)$$

It is now a simple matter of substitution to derive (1.1) from (1.2), (1.3), (1.4), (1.5) and (1.6). Care must be taken not to subtract transfinite numbers, but this causes no difficulty when we note that A, B_1 and C have finite Betti numbers. Thus Theorem 1 is established.

2. *Removal of an Open Set.*—The following theorem is an almost direct corollary of the Alexander duality theorem.

THEOREM 2. *Let S be an n-manifold with the Betti numbers of an n-sphere. Let D be a closed proper sub-set of the points of S, and K a point set on D which is open on S. Then*

$$R_i(D - K) = R_i(D) + R_{n-i-1}(K), i = 0, 1, \ldots, n - 1, \quad (2.1)$$

where $R_i(D)$ and $R_i(D - K)$ are Vietoris Betti numbers.

Proof.—Since K and $(S - D)$ are open sets having no points in common, neither contains a limit point of the other. From this fact it can be shown easily that

$$R_i[K + (S - D)] = R_i(K) + R_i(S - D), \quad (2.2)$$

since topological Betti numbers are meant. Now we write the Alexander duality theorem for the closed sets D and $(D - K)$:

$$R_i(D) = R_{n-i-1}(S - D) + \delta^i{}_0 - \delta^i{}_{n-1} \quad (2.3)$$

$$R_i(D - K) = R_{n-i-1}(K) + R_{n-i-1}(S - D) + \delta^i{}_0 - \delta^i{}_{n-1}, \quad (2.4)$$

where (2.4) is written with the aid of (2.2). By substituting from (2.3) in (2.4) we obtain (2.1), and the theorem is proved.[9]

[1] NATIONAL RESEARCH FELLOW.

[2] A. B. Brown, this number of these PROCEEDINGS.

[3] L. Vietoris, *Math. Ann.*, **97**, 454–472 (1927); **101**, 219–225 (1929).

[4] S. Lefschetz, *Annals of Math.*, (2) **29**, 232–254 (1928).

[5] J. W. Alexander, *Trans. Amer. Math. Soc.*, **23**, 333–349 (1922).

[6] P. Alexandroff, *Compt. Rend.*, **184**, 425–428 (1927).

[7] S. Lefschetz, these PROCEEDINGS, **13**, 614–622, 805–807 (1927).

[8] S. Lefschetz, forthcoming Colloquium Publication, Topology, Ch. VII.

[9] Added in proof-sheets. Another proof has been found, which is longer, but depends only on the Alexander theorem. It will appear in *Ann. Math.*