In the preparation of the material and the interpretation of the results, I have had the coöperation of Dr. Mary B. Stark of the Flower Hospital Staff and the assistance of Miss A. K. Marshall of the Bell Telephone Laboratories.

The photographs which accompany this paper are sufficiently described in the titles and need not be specifically dealt with here.

[1] Lucas, "An Introduction to Ultra-Violet Metallography," *Trans. Am. Inst. Mining and Metallurg. Eng.*, February, 1926.

[2] Lucas, "A Resumé of the Development and Application of High Power Metallography and the Ultra-Violet Microscope," **1,** *Proc. Int. Cong. Test. Materials,* Amsterdam, Holland, 1927.

[3] Lucas, "Photomicrography and Its Application to Mechanical Engineering," *Mech. Eng.*, **50,** pp. 205–212, March, 1928.

[4] Köhler, "Microphotographic Examinations with Ultra-Violet Light," *Zeit. Wissensch. Mikroskopie und für Mik. Tech.*, **21,** 1904, pp. 129–165 and 273–304.

A FUNDAMENTAL THEOREM ON ONE-PARAMETER CONTINUOUS GROUPS OF PROJECTIVE FUNCTIONAL TRANSFORMATIONS[1]

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Let L^x, L^x_s, L^x_1, L^1_s be real bounded integrable functions of the real variables x and s as indicated ($a \leq x, s \leq b$) and let us denote Riemann integration on (a, b) by the repetition of a superscript and subscript in the same term unless one of them is enclosed in a parenthesis.

The regular infinitesimal projective transformation in function space

$$\varphi^x = \bar{\varphi}^x + \delta t[L^x\bar{\varphi}^x + L^x_s\bar{\varphi}^s + L^x_1 - \bar{\varphi}^x L^1_s\bar{\varphi}^s] \tag{1}$$

will generate by continuous application a family of projective functional transformations[2]

$$\bar{\varphi}^x(t) = \frac{K^x(t)\varphi^x + K^x_s(t)\varphi^s + K^x_1(t)}{K^1_s(t)\varphi^s + K^1_1(t)} \tag{2}$$

where $\bar{\varphi}^x(t)$ satisfies the integro-differential system

$$\left.\begin{aligned} \frac{\partial\bar{\varphi}^x(t)}{\partial t} &= L^x\bar{\varphi}^x(t) + L^x_s\bar{\varphi}^s(t) + L^x_1 - \bar{\varphi}^x(t)L^1_s\bar{\varphi}^s(t) \\ \bar{\varphi}^x(0) &= \varphi^x \end{aligned}\right\} \tag{3}$$

Dines[3] has shown that in order for $\bar{\varphi}^x(t)$ in (2) to satisfy (3) the following equations hold

$$\left.\begin{aligned}
'K_s^x &= L^x K_s^x + L_s^x K^{(s)} + L_u^x K_s^u + L_1^x K_s^1 \\
'K_1^x &= L^x K_1^x + L_u^x K_1^u + L_1^x K_1^1 \\
'K_s^1 &= L_s^1 K^{(s)} + L_u^1 K_s^u \\
'K_1^1 &= L_u^1 K_1^u \\
'K^x &= L^x K^x
\end{aligned}\right\} \tag{4}$$

$$K^x(0) = K_1^1(0) = 1, \qquad K_s^x(0) = K_1^x(0) = K_s^1(0) = 0, \tag{5}$$

where the primes indicate differentiation with respect to t.

If (α, β) represent any set of indices which the K's may have, assume a solution of (4) and (5) as follows

$$K_\beta^\alpha(t) = \sum_{m=0}^{\infty} \frac{t^m}{m!} K_\beta^\alpha\,[m] \tag{6}$$

Substituting in (4) and equating coefficients of powers of t, we obtain the recurrence formulas determining the $K_\beta^\alpha[m]$ successively

$$\begin{aligned}
K_s^x[m+1] &= L^x K_s^x[m] + L_s^x K^{(s)}[m] + L_u^x K_s^u[m] + L_1^x K_s^1[m] \\
K_1^x[m+1] &= L^x K_1^x[m] + L_u^x K_1^u[m] + L_1^x K_1^1[m] \\
K_s^1[m+1] &= L_s^1 K^{(s)}[m] + L_u^1 K_s^u[m] \\
K_1^1[m+1] &= L_u^1 K_1^u[m] \\
K^x[m+1] &= L^x K^x[m]
\end{aligned} \tag{7}$$

and from (5) we have

$$K^x[0] = K_1^1[0] = 1, \quad K_s^x[0] = K_1^x[0] = K_s^1[0] = 0. \tag{8}$$

A dominating series is readily found so that the formal solution (6) is an actual solution of (4) and (5). Hence the transformations (2) generated by (1) from a one-parameter analytic family.

The following recursion formulas will be needed

$$\begin{aligned}
K_s^x[p] &= K^x[m]\,K_s^x[p-m] + K_s^x[m]K^{(s)}[p-m] + K_u^x[m] \\
&\qquad K_s^u[p-m] + K_1^x[m]\,K_s^1[p-m] \\
K_1^x[p] &= K^x[m]\,K_1^x[p-m] + K_u^x[m]\,K_1^u[p-m] + K_1^x[m]K_1^1[p-m] \\
K_s^1[p] &= K_s^1[m]\,K^{(s)}[p-m] + K_u^1[m]K_s^u[p-m] + K_1^1[m]\,K_s^1[p-m] \\
K_1^1[p] &= K_u^1[m]\,K_1^u[p-m] + K_1^1[m]\,K_1^1[p-m] \\
K^x[p] &= K^x[m]\,K^x[p-m]
\end{aligned} \tag{9}$$

which we shall abbreviate

$$K_\beta^\alpha[p] = K_\gamma^\alpha[m]\,K_\beta^\gamma[p-m] \tag{10}$$

For $(p, m) = (0, 0), (1, 0), (1, 1)$ these are easily verified. The general case is proved by induction on p.

It is easily verified that the kernels of the product of two transformations of type (2) generated by the same infinitesimal transformation (1), with parameters t_1 and t_2 is given by

$$P^{\alpha}_{\beta}(t_1,t_2) = K^{\alpha}_{\gamma}(t_1)K^{\gamma}_{\beta}(t_2) \tag{11}$$

where the Greek indices take on exactly the same values as in (10). This is the crux of the proof of the theorem below.

THEOREM. *The one-parameter family of projective functional transformations* (2) *generated by a regular infinitesimal projective functional transformation* (1) *is a one-parameter continuous group.*

[1] A general theory of linear functional equations on a composite range with application to projective functional transformations including a fuller account of the work of this note is to be published elsewhere. These developments are embodied in a California Institute thesis. I am indebted to Prof. A. D. Michal for suggesting these topics and for invaluable suggestions and criticisms.

[2] L. L. Dines, *Trans. Am. Math. Soc.*, **20,** 45 (1919), has given in different notation the inversion and group properties for transformation of type (2) and has shown the existence of the one-parameter family satisfying (4) and (5). G. Kowalewski, *Wien. Ber.*, **120,** 1435, has given the name "regular infinitesimal projective functional transformation" to (1).

[3] Loc. cit., p. 59; see also, I. A. Barnett, *Bull. Am. Math. Soc.*, **36,** 273 (1930)

A SPECIAL TYPE OF UPPER SEMI-CONTINUOUS COLLECTION[1]

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1. *Introduction.*—The object of this paper is to show, in the final section, an application of the special type of upper semi-continuous collection of continua[2] which is discussed in § 3. Before doing so, we shall prove certain theorems concerning upper semi-continuous collections in general.

2. *G-Maps on a Cactoid.*—R. L. Moore has shown that an upper semi-continuous collection of continua which fills up a sphere is topologically equivalent to a cactoid.[3] Since a plane is topologically equivalent to a sphere minus a point, this theorem can be extended to the case where the collection fills up a plane, in which case the collection is topologically equivalent to a cactoid minus a non-cut point.

If then G is an upper semi-continuous collection of continua which fills up a sphere (or plane) S, a given correspondence T between the elements of G and the points of a cactoid (or cactoid minus a non-cut point) Σ, affects a kind of "mapping" of the points of S upon the points of Σ. To define this "mapping" more precisely:

Let F be any subset of S, and let G_F be the collection of elements of G