$$Se^1_{a,l}(c, \theta) = \sum_n{}' D^l_n \cos n\theta, \tag{32}$$

$$So^1_{a,l}(c, \theta) = (\sin \theta)^{-2a-1} \sum_n{}' F^l_{n+1} \sin (n + 1)\theta. \tag{33}$$

Associated with the $So^1_{a,l}$ are the functions $Ro^1_{a,l}$ which by analogy with the $Re^1_{a,l}$ are most readily defined by

$$(-1)^a k_l Ro^1_{a,l}(c, z) = (z^2 - 1)^{-a} \int_{-1}^{1} e^{iczt} So^1_{a,l}(t)dt, \tag{34}$$

$$Ro^1_{a,l} = (z^2 - 1)^{-a}(cz)^{a-1/2} \sum{}' g^l_n J_{n-a+1/2}(cz), \tag{35}$$

$$\frac{(i)^n\sqrt{2\pi}\Gamma(n - 2a + 1)}{\Gamma(n + 1)} f^l_n = k_l g^l_n. \tag{36}$$

The determination of the characteristic values b and the expansion coefficients as functions of the parameters a and c is essential to the complete definition of the functions, and a detailed account of this portion of the investigation will be given elsewhere.

[1] Bateman, *Partial Differential Equations of Mathematical Physics*, p. 440 et seq. For a rather complete account of previous work on the subject and a bibliography, see Strutt, *Lame'sche-Mathieusche-und verwandte Funktionen in Physik und Technik*, in the collection *Ergebnisse der Mathematik*, Springer, 1932.

ADDITION FORMULAE FOR SPHEROIDAL FUNCTIONS

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The functions developed by Stratton[1] in the preceding paper are of considerable importance in the study of wave motion in elliptic cylinder and in spheroidal coördinates. By their means a large number of diffraction problems can be studied: the scattering of waves from a thin strip, from a rod or a disc, the diffraction of waves through a slit or through a circular aperture, the scattering of electron waves from a diatomic molecule, etc. Before these problems can be solved, however, a number of addition formulae must be obtained, relating the spheroidal functions to the other known solutions of the wave equation. Some of these formulae are developed below.

1. *Elliptic Cylinder Coördinates.*—In the elliptic cylinder coördinates, $x = (d/2)\cos\varphi \cosh\mu$, $y = (d/2)\sin\varphi \sinh\mu$, the solutions of the wave equation which are everywhere finite are,

$$\psi^1_m(k; \varphi, \mu) = Se^1_{-1/2,\, m}(c, \cos\varphi) Re^1_{-1/2,\, m}(c, \cosh\mu) \qquad (1)$$
$$\chi^1_m(k; \varphi, \mu) = So^1_{-1/2,\, m}(c, \cos\varphi) Ro^1_{-1/2,\, m}(c, \cosh\mu)$$

where $c = (kd/2)$, and where the functions Se, Re, So and Ro are the solutions of

$$(1 - z^2)y'' - zy' + (b - c^2z^2)y = 0 \qquad (2)$$

which are defined by Stratton. The constant c is equal to $(\pi d/\lambda)$, where d is the interfocal distance of the ellipses and λ is the wave-length. The separation constant b has characteristic values b_m corresponding to the various values of m for the functions Se and Re, and has the values b'_m for the functions So, Ro.

The functions[2] can be expanded in the following series:

$$Se^1_{-1/2,\, m}(c, \cos u) = \sum_n{}' D^m_n \cos(nu), \quad \sum_n{}' D^m_n = 1$$
$$Re^1_{-1/2,\, m}(c, \cosh\mu) = \sqrt{\pi/2} \sum_n{}' i^{n-m} D^m_n J_n(c \cosh\mu) \qquad (3)$$
$$So^1_{-1/2,\, m}(c, \cos u) = \sum_n{}' F^m_n \sin(nu), \quad \sum_n{}' nF^m_n = 1$$
$$Ro^1_{-1/2,\, m}(c, \cosh\mu) = \sqrt{\pi/2}\, \tanh\mu \sum_n{}' i^{n-m} nF^m_n J_n(c \cosh\mu)$$

where the prime indicates that the summation is for all even values of n if m is even, for all odd values of n if m is odd.

One important addition formula is the expansion of a plane wave in terms of a series of these functions. If the plane wave have a direction of propagation at an angle u to the x axis, then its form is $\exp(ikY)$, where $k = (2\pi/\lambda)$ and

$$Y = x \cos u + y \sin u = (d/2)(\cos u \cos\varphi \cosh\mu + \sin u \sin\varphi \sinh\mu)$$

Since the ψ's and χ's together form a complete, orthogonal set of characteristic functions, the plane wave can be expanded in the series

$$\exp(ikY) = \sum_m [A_m \psi_m(k;\ \varphi, \mu) + B_m \chi_m(k;\ \varphi, \mu)]$$

where the coefficients A_m and B_m are functions of u.

Due to the symmetry between u and φ in Y, there must be symmetry between u and φ in the series, and therefore the dependence of A_m and B_m on u must be given by the equations

$$A_m = a_m\, Se^1_{-1/2,\, m}(c, \cos u), \qquad B_m = b_m\, So^1_{-1/2,\, m}(c \cos u)$$

where a_m and b_m are independent of u.

To find a_m and b_m, we multiply both sides of the equation for $\exp(ikY)$ by $Se^1_{-1/2,\, m}(c, \cos\varphi)$, and integrate over φ from 0 to 2π. Then we set u small and equate coefficients of powers of u on both sides of the equation.

For the coefficient of the zeroth power of u we have (since $Se(c, \cos u)$ goes to unity as u goes to zero)

$$\int_0^{2\pi} \exp(ic \cos\varphi \cosh\mu) Se^1{}_{-1/2,\,m}(c, \cos\varphi) d\varphi = a_m N_m Re^1{}_{-1/2,\,m}(c, \cosh\mu)$$

where

$$N_m = \int_0^{2\pi} [Se^1{}_{-1/2,\,m}(c, \cos z)]^2 dz = \sum_n{}'(1 + \delta_{on})\pi(D_n^m)^2.$$

By the integral definition of Re^1 given by Stratton in his equation (21) we see that the left side of the equation is $2\sqrt{2\pi}\, i^m\, Re^1{}_{-1/2,\,m}(c, \cosh\mu)$, which shows that the coefficients a_m have the values $(\sqrt{8\pi}\, i^m/N_m)$.

For the coefficients b_m, we multiply by $So^1{}_{-1/2,\,m}(c, \cos\varphi)$ and integrate over φ (remembering that $So(c, \cos u) \longrightarrow u$ as $u \longrightarrow 0$). The coefficient of the first power of u is

$$icu \sinh\mu \int_0^{2\pi} \sin\varphi \exp(ic \cos\varphi \cosh\mu) So^1{}_{-1/2,\,m}(c, \cos\varphi) d\varphi$$
$$= b_m N'_m Ro^1{}_{-1/2,\,m}(c, \cosh\mu) u$$

where

$$N'_m = \int_0^{2\pi} [So^1{}_{-1/2,\,m}(c, \cos z)]^2 dz = \sum_n{}'\pi(F_n^m)^2.$$

From the integral relation (34) in Stratton's paper, we see that the left side is just $2\sqrt{2\pi}\, i^m\, u\, Ro^1{}_{-1/2,\,m}(c, \cosh\mu)$, so that $b_m = (\sqrt{8\pi}\, i^m/N'_m)$.

The complete addition formula is then

$$\exp(ikY) = \sqrt{8\pi} \sum_n i^m [(1/N_m) Se^1{}_{-1/2,\,m}(c, \cos u) \psi^1{}_m(k;\ \varphi, \mu)$$
$$+ (1/N'_m) So^1{}_{-1/2,\,m}(c, \cos u) \chi^1{}_m(k;\ \varphi, \mu)] \qquad (4)$$

and the general integral equation for the elliptic cylinder functions is

$$\int_0^{2\pi} \exp(ikY)\, Se^1{}_{-1/2,\,m}(c, \cos u) du = \sqrt{8\pi}\, i^m\, \psi^1_m(k;\ \varphi, \mu), \qquad (5)$$

with a similar expression for So^1 and $\chi^1{}_m$.

The integral expression for a cylindrical wave[3] is

$$\int_0^{2\pi} \exp(ikY) \cos(mu) du = 2\pi i^m \cos(m\Phi) J_m(k\rho),$$

where $Y = x \cos u + y \sin u = \rho \cos(\Phi - u)$. By utilizing the Fourier series expansion of Se^1 given in (3), we obtain the following relations between the circular cylinder functions and the elliptic cylinder functions:

$$\psi^1{}_m(k;\ \varphi, \mu) = \sqrt{\pi/2} \sum_n{}' D_n^m \cos(m\Phi) J_m(k\rho) \qquad (6)$$
$$\cos(m\Phi) J_m(k\rho) = \sqrt{2\pi}(1 + \delta_{om}) \sum_n{}'(1/N_n) D_m^n \psi^1{}_n(k;\ \varphi, \mu)$$

with similar expressions for the function $\chi^1{}_m$, involving $\sin(m\Phi)$ instead of $\cos(m\Phi)$.

Lastly, it can be shown[4] that the expression for a cylindrical outgoing wave from the point (x', y') is

$$H_0^{(1)}(kP) = (2/\pi)\int_0^{\pi/2 - i\infty} \exp[\pm ik(Y - Y')]du \qquad (7)$$

where $P^2 = (x - x')^2 + (y - y')^2$, $Y' = x' \cos u + y' \sin u$, and the sign of the exponent in the integral is taken positive if $x > x'$, negative if $x < x'$.

From equation (28) of Stratton's paper, relating Se^1 and Re^3, we can obtain

$$\int_0^{\pi/2 - i\infty} \exp(\pm ikY)Se^1{}_{-1/2,\,m}(c, \cos u)du = \sqrt{\pi/2}\; i^m\, \psi^3{}_m(k; \varphi, \mu) \qquad (8)$$

where

$$\psi^3{}_m(k; \varphi, \mu) = Se^1{}_{-1/2,\,m}(c, \cos\varphi)Re^3{}_{-1/2,\,m}(c, \cosh\mu).$$

The positive sign in the exponential is taken if $-\pi/2 < \varphi < \pi/2$, and the negative sign if $\pi/2 < \varphi < 3\pi/2$. A similar relation holds between So^1 and $\chi^3{}_m(k; \varphi, \mu)$.

From (8), by methods similar to those used in deriving (4), we have for the expansion of a cylindrical wave radiating from the point (x', y') in terms of elliptic cylinder functions,

$$H_0^{(1)}(kP) = 4\sum_m [(1/N_m)\psi^1{}_m(k; \varphi, \mu)\psi^3{}_m(k; \varphi', \mu') + (1/N'_m)\chi^1{}_m(k; \varphi, \mu)\chi^3{}_m(k; \varphi', \mu')]$$

when $\mu' > \mu$. When $\mu' < \mu$, φ and μ are interchanged with φ' and μ'.

2. *Prolate Spheroidal Coördinates.*—In the prolate spheroidal coördinates, $z = (d/2)\cosh\mu \cos\vartheta$, $x = (d/2)\sinh\mu \sin\vartheta \cos\varphi$, $y = (d/2)\sinh\mu \sin\vartheta \sin\varphi$, the finite solutions of the wave equation are $\cos(m\varphi)$ or $\sin(m\varphi)$ times $\psi^1{}_{mn}(k; \vartheta, \mu)$, where

$$\psi^1{}_{mn}(k; \vartheta, \mu) = (\sinh\mu \sin\vartheta)^m\, Se^1{}_{mn}(c, \cos\vartheta)\, Re^1{}_{mn}(c, \cosh\mu), \qquad (10)$$

where the functions Se and Re are the solutions of

$$(1 - z^2)y'' - 2(m + 1)zy' + (b - c^2z^2)y = 0 \qquad (11)$$

which are defined by Stratton in the preceding paper. The constant $c = \pi d/\lambda = dk/2$, as before. The functions are given by the series:

$$Se^1{}_{mn}(c, z) = \sum_k{}' d_k^{mn} T_k^m(z),\; \sum_k{}' d_k^{mn} \frac{(2m + k)!}{k!} = 2^m m! \qquad (12)$$

$$Re^1_{mn}(c, z) = \sqrt{\pi/2}\,\frac{(cz)^{-m-1/2}}{2^m m!}\sum_k{}' i^{m-k}\,\frac{(2m+k)!}{k!}\,d_k^{mn}\,J_{k+m+1/2}(cz).$$

The plane wave in three dimensions whose direction of propagation with respect to the z axis is defined by the angles ω, α is $\exp(ikX)$, where

$$\begin{aligned} X &= z\cos\omega + x\sin\omega\cos\alpha + y\sin\omega\sin\alpha \\ &= (d/2)[\cos\omega\cos\vartheta\cosh\mu + \sin\omega\sin\vartheta\sinh\mu\cos(\varphi-\alpha)]. \end{aligned}$$

This must be expressible as a series in the solutions (10), and due to the symmetry between ω and ϑ in X, we must have

$$\exp(ikX) = \sum_{m,n} a_{mn}\cos[m(\varphi-\alpha)]\sin^m\omega\, Se^1_{mn}(c,\cos\omega)\,\psi^1_{mn}(k;\vartheta,\mu). \quad (13)$$

To determine the coefficients a_{mn}, we first expand $\exp(ikX)$ as a Fourier series in $(\varphi - \alpha)$, setting down only the first terms in powers of $\sin\omega$,

$$\exp(ikX) = \exp(ic\cosh\mu\cos\vartheta)\sum_m\frac{\cos[m(\varphi-\alpha)]}{(1+\delta_{om})}\left[\frac{(ic)^m}{2^{m-1}m!}(\sin\omega\sin\vartheta\sinh\mu)^m + \ldots\right].$$

Multiplying both sides of (13) by $\cos[m(\varphi - \alpha)]$, integrating over $(\varphi - \alpha)$ and expanding the $Se^1_{mn}(c, \cos\omega)$ on the right side in powers of $\sin\omega$, gives

$$\pi\exp(ic\cosh\mu\cos\vartheta)\left[\frac{(ic)^m}{2^{m-1}m!}(\sin\omega\sin\vartheta\sinh\mu)^m + \ldots\right] = (1+\delta_{om})\pi\sum_n a_{mn}\psi_{mn}(k;\vartheta,\mu)[\sin^m\omega + \ldots].$$

We next multiply both sides by $\sin^m\vartheta Se^1_{mn}(c, \cos\vartheta)d(\cos\vartheta)$, and integrate over ϑ from 0 to π, then equate the coefficients of the lowest power of $\sin\omega$ on each side of the equation;

$$\frac{\pi(ic)^m}{2^{m-1}m!}\sinh^m\mu\int_0^\pi\exp(ic\cosh\mu\cos\vartheta)\sin^{2m+1}\vartheta Se^1_{mn}(c,\cos\vartheta)d\vartheta = (1+\delta_{om})\pi N_{mn}\,a_{mn}\sinh^m\mu\, Re^1_{mn}(c,\cosh\mu)$$

where

$$N_{mn} = \int_0^\pi[\sin^m\vartheta Se^1_{mn}(c,\cos\vartheta)]^2\sin\vartheta d\vartheta = \sum_k{}'\frac{2(2m+k)!}{(2m+2k+1)k!}(d_k^{mn})^2.$$

By the use of the equation (21) given by Stratton, relating Se and Re, we finally obtain the value of a_{mn}, and the plane wave addition formula turns out to have the form,

$$\exp(ikX) = 2\sum_{m,n}\cos[m(\varphi-\alpha)]\frac{c^m i^{m+n}}{N_{mn}}\sin^m\omega Se^1_{mn}(c,\cos\omega)\psi^1_{mn}(k;\vartheta,\mu). \quad (14)$$

The general integral form for the function ψ is therefore,

$$\int_0^{2\pi} d\alpha \int_0^{\pi} d\varphi \exp(ikX) \sin^m\vartheta Se^1_{mn}(c, \cos\vartheta) {\cos \atop \sin}(m\alpha) = 4\pi c^m i^{m+n} {\cos \atop \sin}(m\varphi)\, \psi^1_{mn}(k; \vartheta, \mu). \quad (15)$$

If the integration over ω be from 0 to $(\pi/2) - i\infty$, then instead of $4\pi\psi^1$ on the right-hand side, we will have $2\pi\psi^3$, everything else being unchanged. Here

$$\psi^3_{mn}(k; \vartheta, \mu) = (\sin\vartheta \sinh\mu)^m \; Se^1_{mn}(c, \cos\vartheta) \; Re^3_{mn}(c, \cosh\mu).$$

It can be shown that the integral relation[5] for spherical waves is

$$\int_0^{2\pi} d\alpha \int_0^{\pi} d\omega \exp(ikX) \sin^{m+1}\omega T^m_n(\cos\omega) {\cos \atop \sin}(m\alpha) = 4\pi i^{m+n} {\cos \atop \sin}(m\varphi) \sin^m\Theta T^m_n(\cos\Theta)\sqrt{\pi/2kr}\; J_{m+n+1/2}(kr) \quad (16)$$

$$\text{where } X = z\cos\omega + x\sin\omega\cos\alpha + y\sin\omega\sin\alpha = r[\cos\omega\cos\Theta + \sin\omega\sin\Theta\cos(\varphi - \alpha)].$$

By using equations (12), (15) and (16), we can obtain relationships between the spheroidal waves and spherical waves[6]

$$c^m\psi^1_{mn}(k; \vartheta, \mu) = \sum_k{}' \; i^{k-n} d^{mn}_k \sin^m\Theta \; T^m_k(\cos\Theta) \sqrt{\pi/2kr}\; J_{m+k+1/2}(kr) \quad (17)$$

$$\sin^m\Theta \; T^m_n(\cos\Theta) \sqrt{\pi/2kr}\; J_{m+n+1/2}(kr) = \frac{2c^m n!}{(2n+2m+1)(2m+n)!} \sum_k{}' \frac{i^{k-n}}{N_{kn}} d^{mk}_n \psi^1_{mk}(k; \vartheta, \mu).$$

Finally, by methods similar to those used in obtaining (9), the expansion of the spherical wave going out from the point (x', y', z') is

$$\frac{e^{ikR}}{R} = \frac{ik}{2\pi}\int_0^{2\pi} d\alpha \int_0^{\pi} d\omega e^{ik(X-X')}\sin\omega = 2ik \sum_{m,n} (2 - \delta_{om}) \frac{c^{2m}}{N_{mn}} \cos[m(\varphi - \varphi')]\psi^1_{mn}(k; \vartheta, \mu)\, \psi^3_{mn}(k; \vartheta', \mu') \quad (18)$$

where $R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$. This series holds for $\mu' > \mu$; when $\mu' < \mu$, we interchange ϑ, μ with ϑ', μ'.

3. *Oblate Spheroidal Coördinates.*—In the oblate spheroidal coördinates $z = (d/2)\sinh\mu\cos\vartheta$, $x = (d/2)\cosh\mu\sin\vartheta\cos\varphi$, $y = (d/2)\cosh\mu\sin\vartheta\sin\varphi$, the finite solutions of the wave equation are ${\cos \atop \sin}(m\varphi)\; \psi^1_{mn}(k; \vartheta, \mu)$, where

$$\psi^1_{mn}(k; \vartheta, \mu) = (\cosh\mu\sin\vartheta)^m \; Se^1_{mn}(ic, \cos\vartheta) \; Re^1_{mn}(ic, -i\sinh\mu). \quad (19)$$

The addition formulae for these functions have the same form as the equations (14), (17) and (18) for prolate spheroidal functions, the only difference being that the definition of ψ^1_{mn} is now (19) instead of (10), and that c is changed to ic in the functions Se.

[1] Stratton, *Proc. Nat. Acad. Sci.*, **1**, 51–56 (1935).

[2] The functions $Se^1_{-1/2,m}(c, \cos u)$ are of course proportional to the Mathieu functions $ce_m(u)$, and the So's to the functions $se_m(u)$.

[3] Whittaker and Watson, *Modern Analysis*, page 396.

[4] Watson, *Theory of Bessel Functions*, page 180.

[5] Whittaker and Watson, *Modern Analysis*, page 398.

[6] Relations can also be obtained between spheroidal waves and cylindrical waves.

ON THE BENDING OF ELECTROMAGNETIC MICRO-WAVES BELOW THE HORIZON

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1. An interesting phase in the development of the modern radio technique are the experiments conducted during the last few years with very short wave-lengths. Marchese Marconi[1] reported about an extensive series of successful radio connections over distances up to 260 km., in which waves of from 50 cm. to 60 cm. were used, concentrated with the help of a parabolic reflector. Clavier and Gallant[2] went even to still shorter waves of only 17.4 cm. which they sent over a distance of 61 km. also concentrating them with a reflector of 3.8 m. in diameter. The most remarkable feature of Marchese Marconi's results is that the distances covered by him exceed several times the range of rectilinear visibility from the sending station.

The memory is still fresh of the great surprise which was caused among physicists by the unusually long range of long wave radio-reception. The explanation of these puzzling facts about long waves was traced, in the meantime,[3] to the influence of the Kennelly-Heaviside layer of the upper atmosphere, and the question, naturally, arises to what extent atmospheric influences are responsible for the phenomena observed by Marchese Marconi with micro-waves. The first step in answering this question must be an investigation of how much bending is to be expected from the point of view of the wave theory *completely neglecting the atmosphere.* Such an investigation is the subject of this paper.

The simple method which we propose is based on Huyghens' principle and treats the surface of the earth as a perfectly absorbing screen. As far as we know, it was not used heretofore and there are good reasons for this: In the case of long waves, the properties of the soil play an important