α, whenever Σu_{mn} is summable $(N; c)$ and satisfies (13), *and should furthermore approach the value to which Σu_{mn} is summable as $\alpha \rightarrow \alpha_0$, are that the convergence factors $f_{mn}(\alpha)$ satisfy the conditions of Theorem I and the further conditions* (A_1), (C_1), (C_2), (E_1) *and* (E_2).

Since Cesàro means of any order whose real part is positive are special cases of the Nörlund means used in this note, our theorems include as special cases convergence factor theorems[2] for double series summable (C).

[1] C. N. Moore, "On Convergence Factors for Series Summable by Nörlund Means," these PROCEEDINGS, **21**, 263–266 (1935).

[2] C. N. Moore, "On Convergence Factors in Multiple Series," *Trans. Amer. Math. Soc.*, **29**, 227–238 (1927). Cf. also Abstract No. 40-1-17, *Bull. Amer. Math. Soc.*, **40**, 32–33 (1934).

FUNCTIONAL DIFFERENTIAL EQUATIONS AND INEQUALITIES

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Let us first try to find the minimum value of the integral

$$\int_0^{2\pi} [f'(x) + mf(x+\pi) + e(x)]^2 dx \qquad (1)$$

where $f(x)$ is a uniform function of period 2π which is integrable and such that

$$\int_0^{2\pi} [f(x)]^2 dx = 1.$$

Replacing $f(x)$ by $f(x) + h(x)$ we have

$$\begin{aligned} &\int_0^{2\pi} [f'(x) + mf(x+\pi) + e(x) + h'(x) + mh(x+\pi)]^2 dx \\ &= \int_0^{2\pi} [f'(x) + mf(x+\pi) + e(x)]^2 dx + \int_0^{2\pi} [h'(x) + mh(x+\pi)]^2 dx \\ &\quad + 2\int_0^{2\pi} h'(x) [f'(x) + mf(x+\pi) + e(x)] dx \qquad (2) \\ &\quad 2m\int_0^{2\pi} h(x+\pi) [f'(x) + mf(x+\pi) + e(x)] dx. \end{aligned}$$

Now if $e(x)$, $f(x)$ and $h(x)$ all have the period 2π,

$$\int_0^{2\pi} h'(x) [f'(x) + mf(x+\pi) + e(x)] dx = -\int_0^{2\pi} h(x) [f''(x) + mf'(x+\pi) + e'(x)] dx,$$

also

$$m\int_0^{2\pi} h(x+\pi) [f'(x) + mf(x+\pi) + e(x)] dx$$

$$= m\int_0^{2\pi} h(x)[f'(x+\pi) + mf(x) + e(x+\pi)]dx.$$

Choosing $f(x)$ so that

$$f''(x) + mf'(x+\pi) + e'(x) - m[f'(x+\pi) + mf(x) + e(x+\pi)] = kf(x),$$

where k is a positive constant and noting that since

$$\int_0^{2\pi}[f(x) + h(x)]^2dx = 1, \int_0^{2\pi}[f(x)]^2dx = 1,$$

we have

$$-2k\int_0^{2\pi} f(x)h(x)dx = k\int_0^{2\pi}[h(x)]^2dx,$$

the right-hand side of (2) becomes

$$\int_0^{2\pi}[f'(x) + mf(x+\pi) + e(x)]^2dx + \int_0^{2\pi}[h'(x) + mh(x+\pi)]^2dx + k\int_0^{2\pi}[h(x)]^2dx.$$

To find the minimum value of the integral (1) we have then to solve the differential equation

$$f''(x) - (m^2 + k)f(x) = me(x+\pi) - e'(x)$$

when the supplementary conditions are $f(x + 2\pi) = f(x)$, $\int_0^{2\pi}[f(x)]^2dx = 1$. If $e(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \ldots + b_1 \sin x + b_2 \sin 2x + \ldots$ it is readily seen that

$$\text{Min.} \int_0^{2\pi}\left[f'(x) + mf(x+\pi) + e(x)\right]^2 dx = mk^2\left[\frac{2a_0^2}{(m^2+k)^2} + \frac{a_1^2 + b_1^2}{(m^2 + l^2 + k)^2} + \ldots\right]$$

where k is given by the equation

$$1 = \pi\left[\frac{2m^2a_0^2}{(m^2+k)^2} + \frac{(m^2 + l^2)(a_1^2 + b_1^2)}{(m^2 + l^2 + k)^2} + \ldots\right]$$

In the particular case when $e(x) = a \cos sx + b \sin sx$ and $\pi(a^2 + b^2) \geqq s^2 + m^2$ we have the inequality

$$\int_0^{2\pi}\left[f'(x) + mf(x+\pi) + a\cos sx + b\sin sx\right]^2 dx \geqq \left[\pi^{\frac{1}{2}}(a^2 + b^2)^{\frac{1}{2}} - (s^2 + m^2)^{\frac{1}{2}}\right]^2$$

which may be written in the alternative form

$$\left|\int_0^{2\pi}[F'(x) + mF(x+\pi)]\cos(sx)dx\right| \frac{2}{\pi^{1/2}} \leq (m^2 + s^2)^{\frac{1}{2}}\left[\int_0^{2\pi}\{F(x)\}^2dx\right]^{\frac{1}{2}}$$

$$+ (m^2 + s^2)^{-\frac{1}{2}} \left[\int_0^{2\pi} [F(x)]^2 dx \right]^{-\frac{1}{2}} \int_0^{2\pi} \left[F'(x) + mF(x + \pi) \right]^2 dx,$$

$F(x)$ being any periodic differentiable function for which the integrals exist, the period being 2π.

If $p(x)$ is a positive function integrable throughout the range $(0, 2\pi)$ the problem of finding a minimum value for the integral

$$\int_0^{2\pi} \left\{ \sum_{r=0}^{n} \left[u_r(x) f^{(r)}(x) + v_r(x) f^{(r)}(x + \pi) + e_r(x) \right] \right\}^2 p(x) dx$$

when $f(x)$ is a periodic integrable function for which*

$$\int_0^{2\pi} [f(x)]^2 dx = 1$$

generally leads to a functional differential equation for $f(x)$ and an associated inequality. It is only occasionally that the equation for $f(x)$ reduces to a differential equation but it may sometimes reduce to a simple functional equation or difference equation. The analysis is easily extended by replacing the substitution $z = x + \pi$ by some other substitution $z = s(x)$ which transforms the range of integration into itself by a one to one correspondence.** The equation derived from the variation problem is then an iterative differential equation involving functions of x, $s(x)$ and $s(s(x))$.

* This condition may be replaced by some other condition such as $\int_0^{2\pi} f(x) g(x) dx = 1$.

** Another generalization is obtained by considering a summation over all integral values of n of $p_n [u_n f_n + v_n f_{n+1} + e_n]^2$, when the sum of f_n^2 is unity,and $P_n > 0$.

A COMPLETE CHARACTERIZATION OF THE DERIVATIVE OF A POLYGENIC FUNCTION[1]

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1. *Introduction.*—A function $w = \varphi(x, y) + i\psi(x, y)$ is called a polygenic function of the complex variable $z = x + iy$ if the real functions φ and ψ are general, that is, are not required to satisfy the Cauchy-Riemann differential equations. The value of the derivative of a polygenic function at $z = z_0$ depends in general not only on the point z_0 but also on the direction θ along which z approaches z_0; that is, $dw/dz = F(z, \theta)$. Thus, the derivative $\gamma = dw/dz$ of a polygenic function w may be regarded as determining a correspondence between lineal elements (x, y, θ) of the z-plane