This group contains 50 operators of order 3 because the extending operator of order 3 could not transform into itself any of its operators of order 5.

[1] Miller, Blichfeldt, Dickson, *Finite Groups*, 1916, p. 127.

TRIHORNOMETRY: A NEW CHAPTER OF CONFORMAL GEOMETRY

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I. *Introduction and Definitions.*—The object of this paper is to study the relationships between the conformal invariants of an ordered triplet of curves C_1, C_2 and C_3, which pass through a common point in a common direction. This configuration is termed a *trihorn*. We shall consider only the case where the curves have different curvatures at the common point so that there are no contacts of higher order. A trihorn has six fundamental invariants and these are connected by inequalities and equalities, of which the most important are the formulae (5), (6), (8), (9), (11) and (13). In particular we obtain the analogues of the laws of sines and cosines of ordinary trigonometry.

Let x represent the curvature and y the derivative of curvature with respect to length of arc of a curve C of the trihorn at the common point. The ordered pair of curves (C_i, C_j) of our trihorn form a horn angle and the unique absolute conformal invariant

$$M_{ij} = \frac{(x_j - x_i)^2}{y_j - y_i} \tag{1}$$

is called the *measure of the horn angle.* It is noted that the measure is never zero, but it may be infinite. When the measure is infinite, then the horn angle is said to be circular.

The quantity

$$\alpha_{ij} = \frac{\dfrac{y_i - y_k}{x_i - x_k}}{\dfrac{y_j - y_k}{x_j - x_k}} = \frac{(y_i - y_k)(x_j - x_k)}{(x_i - x_k)(y_j - y_k)} \tag{2}$$

is also an absolute conformal invariant. It is to be regarded as a new kind of angle, namely, the angle between the horn (C_i, C_k) and the horn (C_j, C_k). This is called a *dihorn angle.*

We note the obvious relations

$$M_{ji} = -M_{ij},\ \alpha_{ji} = \frac{1}{\alpha_{ij}}.$$

All the invariants are of third order since they involve curvatures and rates of variation of curvatures. (There is also, for three curves, an absolute invariant of second order

$$\Gamma_{1,23} = \frac{x_3 - x_1}{x_2 - x_1}$$

which I shall not use in this paper. This is in fact invariant under all point transformations.)

We thus have six fundamental parts in a trihorn, the three horns (or sides or M's) and the three dihorns (or angles or α's). The following theorems are stated without proof.

II. *Discussion of Special Trihorns.*—THEOREM 1. *If two measures of a trihorn are infinite, then all the measures are infinite and the angles are indeterminate. This trihorn is said to be circular.*

THEOREM 2. *If one angle of a trihorn is indeterminate, then all the angles are indeterminate and all the measures are infinite. Thus the trihorn is circular.*

THEOREM 3. *If only* $M_{ij} = \infty$, *then* $M_{jk}M_{ki} < 0$, $\alpha_{jk} = 0$, $\alpha_{ki} = \infty$ *and* $\alpha_{ij} \neq 0, 1, \infty$. *This is called a partially circular trihorn.*

THEOREM 4. *If* $\alpha_{jk} = 0$ *(or* $\alpha_{ki} = \infty$*), then* $\alpha_{ki} = \infty$ *(or* $\alpha_{jk} = 0$*),* $\alpha_{ij} \neq 0, 1, \infty$, $M_{ij} = \infty$ *and* $M_{jk}M_{ki} < 0$. *Thus the trihorn is partially circular.*

THEOREM 5. *For the partially circular trihorn of theorems* 3 *and* 4, *we have the unique equality*

$$\alpha^2_{ij}\, M_{ki} + M_{jk} = 0. \tag{3}$$

A trihorn which is neither circular nor partially circular is termed a non-circular trihorn. For a non-circular trihorn, the measures and the angles are determinate, finite, non-zero numbers.

If a non-circular trihorn is such that

$$M_{12} + M_{23} + M_{31} = 0, \tag{4}$$

then it is said to be ***wide-open***. If a non-circular trihorn is not wide-open, then it is termed a general trihorn.

THEOREM 6. *If a non-circular trihorn is wide-open, then each angle is unity. Conversely if an angle of a non-circular trihorn is unity, then all the angles are unity and the trihorn is wide-open.*

If a non-circular trihorn is such that $M_{ij} + M_{jk} = 0$, then it is termed an isosceles trihorn. Obviously no wide-open trihorn can be isosceles, and conversely.

III. *Solutions of General Trihorns.*—If we are given any three parts (except the three angles) of a general trihorn, we proceed to show, in the remaining six theorems, how to find the other three parts. Several of the cases turn out to be *ambiguous*, namely, there may be two solutions or one solution.

In the following, by a *number* we shall mean a determinate, finite, non-zero number. Moreover, when we say that a number is an angle of a trihorn, we shall understand that the number also is distinct from unity.

THEOREM 7. *For three numbers* M_{12}, M_{23}, M_{31}, *to be the measures of a general non-isosceles trihorn, it is necessary and sufficient that no sum* $M_{ij} + M_{jk}$ *be zero, and that*

$$M_{12}M_{23}M_{31}\,(M_{12} + M_{23} + M_{31}) < 0. \tag{5}$$

Then all the angles have two distinct values which are given by the formulae

$$\alpha_{ij} = \frac{-M_{jk}M_{ki} \pm \sqrt{-M_{ij}M_{jk}M_{ki}\,(M_{ij} + M_{jk} + M_{ki})}}{M_{ki}\,(M_{ij} + M_{ki})}. \tag{6}$$

THEOREM 8. *For three numbers* M_{12}, M_{23}, M_{31}, *to be the measures of a general isosceles trihorn, it is necessary and sufficient that a sum* $M_{ij} + M_{jk}$ *be zero, and that all the measures be not equal in absolute value. Then the angles are uniquely determined by the formulae*

$$\alpha_{ij} = \frac{2M_{ij}}{M_{ij} + M_{ki}}, \quad \alpha_{jk} = \frac{M_{ij} - M_{ki}}{2M_{ij}}, \quad \alpha_{ki} = \frac{M_{ij} + M_{ki}}{M_{ij} - M_{ki}}. \tag{7}$$

THEOREM 9. *For three numbers* α_{12}, α_{23}, α_{31}, *to be the angles of a general trihorn, it is necessary and sufficient that*

$$\alpha_{12}\alpha_{23}\alpha_{31} = 1. \tag{8}$$

Then the ratios of the measures are uniquely determined by the formulae

$$\frac{M_{12}}{\alpha_{31}(1 - \alpha_{12})} = \frac{M_{23}}{\alpha_{12}(1 - \alpha_{23})} = \frac{M_{31}}{\alpha_{23}(1 - \alpha_{31})}. \tag{9}$$

This is the analogue of the law of sines in ordinary trigonometry.

THEOREM 10. *The necessary and sufficient conditions that the numbers* M_{ij}, M_{ki} *and* α_{jk} *belong to a general trihorn are*

$$M_{ij}\alpha_{jk} + M_{ki} \neq 0, \; M_{ij}\alpha_{jk}^2 + M_{ki} \neq 0. \tag{10}$$

The remaining parts are uniquely determined by the formulae

$$M_{jk} = -\frac{(M_{ij}\alpha_{jk} + M_{ki})^2}{M_{ij}\alpha_{jk}^2 + M_{ki}}$$

$$\alpha_{ij} = \frac{M_{ij}\alpha_{jk} + M_{ki}}{M_{ij}\alpha_{jk}^2 + M_{ki}} \qquad (11)$$

$$\alpha_{ki} = \frac{1}{\alpha_{ij}\alpha_{jk}}.$$

The first formula of (11) *is the analogue of the law of cosines of ordinary trigonometry.*

In the above theorem we are given two sides and the included angle; in the next theorem we are given two sides and an adjacent angle.

THEOREM 11. *For three numbers* M_{ij}, M_{jk} *and* α_{jk} *to be the measures and angle of a general trihorn, it is necessary and sufficient that*

(*a*) If $M_{ij} + M_{jk} \neq 0$, then

$$M_{jk}^2 + 4M_{ij}M_{jk}\alpha_{jk}\,(1 - \alpha_{jk}) \geqq 0. \qquad (12)$$

The remaining parts have two possible determinations which are given by the formulae

$$\left.\begin{aligned} M_{ki} &= \frac{-2M_{ij}\alpha_{jk} - M_{jk} \pm\sqrt{M_{jk}^2 + 4M_{ij}M_{jk}\alpha_{jk}(1 - \alpha_{jk})}}{2}, \\ \alpha_{ij} &= \frac{M_{ij}\alpha_{jk} + M_{ki}}{M_{ij}\alpha_{jk}^2 + M_{ki}}, \qquad \alpha_{ki} = \frac{1}{\alpha_{ij}\alpha_{jk}}. \end{aligned}\right\} \qquad (13)$$

(*b*) If $M_{ij} + M_{jk} = 0$, then $\alpha_{jk} \neq -1, 1/2$. The remaining parts are uniquely determined by the formulae

$$\left.\begin{aligned} M_{ki} &= -2M_{ij}\alpha_{jk} - M_{jk} \\ \alpha_{ij} &= \frac{M_{ij}\alpha_{jk} + M_{ki}}{M_{ij}\alpha_{jk}^2 + M_{ki}}, \quad \alpha_{ki} = \frac{1}{\alpha_{ij}\alpha_{jk}}. \end{aligned}\right\} \qquad (14)$$

THEOREM 12. *If any three numbers are given, a trihorn can be constructed with two of the numbers as angles and the third as measure. The remaining parts are found by formulae* (8) *and* (9).

We observe that the *special metric* considered in this paper is based on the distance element $ds = dx^2/dy$, that is, on the calculus of variation problem

$$\int \frac{dx^2}{dy} = \int \frac{1}{y'}\,dx = \text{extremum}.$$

It is therefore non-riemannien, and of course non-euclidean.

The fundamental "congruence group" is

$$X = mx + a, \quad Y = m^2y + b;$$

it is induced by the conformal group of the plane $w = f(z)$, and shows the effect on differential elements of third order. The extremals are straight lines (wide-open phenomenon), and transversality is defined by taking half the slope. The angle α of a "right angle" is therefore $\alpha = \frac{1}{2}$. The reciprocal $\alpha = 2$ defines anti-perpendicularity.

With relation to the above integral, angle (dihorn) is properly defined as

$$A = \frac{1}{2} \log \alpha;$$

so that instead of (8) we have, in any trihorn,

$$A_{12} + A_{23} + A_{31} = 0_1 A_{ji} = -A_{ij};$$

but we have preferred to use α so as to make all our formulas algebraic. For related material on conformal geometry see our earlier papers: *Proc. Intern. Congr. Math.*, **2,** 81 (1912); *Proc. Nat. Acad. Sci.* (with G. Comenetz), **22,** 303 (1936); *Science*, **85,** 480 (1937).

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For horn angles of all orders of contact we obtain a non-archimedean geometry. For second order contact the fundamental fifth order invariant includes Mullin's inversive invariant as a special case.

CONTINUOUS RINGS AND THEIR ARITHMETICS

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Introduction. 1. This note continues the analysis of the geometrical systems called *continuous geometries*, discussed in four previous notes of the author.[1] Again only results and outlines of proofs will be given, the details being reserved for subsequent publications.[2]

The main result obtained in A. T. was this: Every continuous geometry L is (lattice) isomorphic to the principal right-ideal lattice $R_{\mathfrak{R}}$ of a suitable regular ring $\mathfrak{R}$, which is uniquely determined by L, up to a (ring) isomorphism.[3] This result establishes a complete characterization (up to a lattice isomorphism) of L, by means of the purely algebraic entity $\mathfrak{R}$, and is satisfactorily complete under this aspect. Yet it is incomplete in so far as it fails to characterize those regular rings $\mathfrak{R}$, which arise in this manner